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# Adhesive High-Level Replacement Systems with Negative Application Conditions

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## Abstract

The goal of this paper is the generalization of basic results for adhesive High-Level Replacement (HLR) systems to adhesive HLR systems with negative application conditions. These conditions restrict the application of a rule by expressing that a specific structure should not be present before or after applying the rule to a certain context. Such a condition influences thus each rule application or transformation and therefore changes significantly the properties of the replacement system. The effect of negative application conditions on the replacement system is described in the generalization of the following results, formulated already for adhesive HLR systems without negative application conditions: Local Church-Rosser Theorem, Parallelism Theorem, Completeness Theorem for Critical Pairs, Concurrency Theorem, Embedding and Extension Theorem and Local Confluence Theorem or Critical Pair Lemma. These important generalized results will support the development of formal analysis techniques for adhesive HLR replacement systems with negative application conditions.

*Keywords:* Negative Application Conditions, Adhesive High-Level Replacement Categories

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## 1 Introduction

Adhesive High-Level Replacement (HLR) categories as introduced in [3] provide a formal method to describe transformation systems. The resulting framework is called adhesive HLR systems. These systems are based on rules that describe in an abstract way how objects in adhesive HLR categories can be transformed. In [3], it is explained moreover how to define application conditions for rules that restrict the application of a rule. Most of the theoretical results in [3] though have been formulated for adhesive HLR systems based on rules without application conditions. These results should thus be generalized to adhesive HLR systems based on rules holding application conditions. The most frequently used kind of application condition is the so-called negative application condition (NAC) as introduced in [4] and used e.g. in [1,5,7,12,15]. It forbids a certain structure to be present before or after

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applying a rule. Therefore at first we concentrate on generalizing the theoretical results formulated for adhesive HLR systems based on rules without application conditions to adhesive HLR systems based on rules holding NACs. Shortly, we will speak about adhesive HLR systems with NACs.

Some important theoretical results for the particular case of graph transformation with NACs have been presented already in [10]. The overall goal is to come up with practical techniques for conflict detection and analysis [11,9] in transformation systems. In practice though most of these results are needed for the instantiation of typed attributed graph transformation systems with application conditions. This more general kind of graph transformation technique is most significant for modeling and metamodeling in software engineering and visual languages. Therefore the availability of all results for typed attributed graph transformation with NACs is an important motivation for the generalization to adhesive HLR systems with NACs. In [3], it has been already proven that such a typed attributed graph transformation system is a valid instantiation of adhesive HLR systems. Moreover results within adhesive HLR systems can be applied to all other instantiations of adhesive HLR systems such as e.g. hypergraph, algebraic signature or specification transformations with NACs.

At first we will generalize the local Church-Rosser property to transformations with NACs. Therefore a new notion of parallel independence and thus also sequential independence is defined between transformations with NACs. This is because it can not only happen that a transformation deletes a structure that is used by the second transformation as considered in the case without NACs. Moreover we should consider the case of the first transformation producing a structure which is forbidden by the second one. This situation is formalized in a new characterization of conflicts between transformations holding NACs. Moreover it is possible to formulate a Parallelism Theorem for transformations with NACs expressing how to summarize a sequence of two sequentially independent transformations into one parallel transformation step with the same effect.

In order to come up with techniques for conflict detection and analysis it is central to describe the theory for the so-called critical pairs. The notion of critical pairs is introduced in the area of term rewriting systems [6] and, later, introduced in the area of graph transformation for hypergraph rewriting [13,14] and then for all kinds of transformation systems fitting into the framework of adhesive HLR categories. Critical pairs describe conflicts in a minimal context. By means of the conflict characterization for transformations with NACs it is possible to define a generalized critical pair notion for adhesive HLR systems with NACs. Critical pairs should fulfill two important properties that were fulfilled in the case of adhesive HLR systems without NACs as well. It should be possible to formulate a Completeness Theorem and a so-called Critical Pair Lemma or Local Confluence Theorem. The Completeness Theorem describes that for every pair of transformations in conflict there exists a critical pair expressing the same conflict in a minimal context. We prove that the critical pair definition for adhesive HLR systems with NACs is complete in this sense.

In order to formulate and prove the Local Confluence Theorem for adhesive HLR systems with NACs we first need to generalize some other important results.

The first one is the Concurrency Theorem and the other ones are the Embedding and Extension Theorems. For the Concurrency Theorem it is explained how to construct a concurrent rule holding NACs from a sequence of rules holding NACs. The Concurrency Theorem with NACs states that the concurrent rule with NACs is applicable with the same result if and only if the rule sequence with NACs is applicable. The construction of the concurrent rule itself is analog to the case without NACs. It is necessary though to translate all NACs occurring in the rule sequence into equivalent NACs on the concurrent rule. Therefore we will use results for application conditions already described in [3] and some new results. Using the Concurrency Theorem it is not too difficult to find an extended condition on the extension morphism in order to generalize also the Embedding and Extension Theorem to adhesive HLR systems with NACs. Finally we are also able to handle critical pairs and local confluence with NACs. However the Local Confluence Theorem for adhesive HLR systems with NACs is not yet fully satisfactory.

## 2 Adhesive HLR systems with NACs

First we repeat the definition for an adhesive HLR category as introduced in [3].

**Definition 2.1** [adhesive HLR category] A category  $\mathbf{C}$  with a morphism class  $\mathcal{M}$  is called an *adhesive HLR category*, if

- (i)  $\mathcal{M}$  is a class of monomorphisms closed under isomorphisms, composition and decomposition,
- (ii)  $\mathbf{C}$  has pushouts (PO) and pullbacks (PB) along  $\mathcal{M}$ -morphisms and  $\mathcal{M}$ -morphisms are closed under pushouts and pullbacks,
- (iii) pushouts in  $\mathbf{C}$  along  $\mathcal{M}$ -morphisms are Van Kampen squares.

**Remark 2.2** Note that all results formulated in this paper will be applicable as well in weak adhesive HLR categories with NACs such as e.g. for Petri Net transformations with NACs.

For an *adhesive HLR category with NACs* we need in addition to an adhesive HLR category without NACs some additional properties on the special morphism classes in the category in order to be able to generalize all results. We distinguish three classes of morphisms, namely  $\mathcal{M}$ ,  $\mathcal{M}'$  and  $\mathcal{Q}$ , and a class of pairs of morphisms  $\mathcal{E}'$ .  $\mathcal{M}$  is a subset of the class of all monomorphisms as given in [3] and the rule morphisms are always in  $\mathcal{M}$ . The non-existing morphism  $q$  in Def. 2.6 for negative application conditions is an element of the morphism set  $\mathcal{Q}$ . For pair factorization in Def. 5.25 in [3] we need moreover the classes  $\mathcal{M}'$  and  $\mathcal{E}'$ . For the case of graph transformation systems with NACs with injective matching we can take  $\mathcal{Q} = \mathcal{M}' = \mathcal{M}$ , where  $\mathcal{M}$  is the set of all graph monomorphisms and  $\mathcal{E}'$  the set of jointly surjective pairs of graph morphisms.  $\mathcal{M}$ ,  $\mathcal{E}'$ ,  $\mathcal{M}'$  and  $\mathcal{Q}$  should have the properties described in the following definition. Note that to each condition a remark is made in which theorem, lemma or definition this condition is needed for the first time.

**Definition 2.3** [adhesive HLR category with NACs] An *adhesive HLR category with NACs* is an adhesive HLR category  $\mathbf{C}$  with special morphism class  $\mathcal{M}$  and in addition three morphism classes  $\mathcal{M}'$ ,  $\mathcal{E}'$  and  $\mathcal{Q}$  with the following properties:

- unique  $\mathcal{E}' - \mathcal{M}'$  pair factorization (see Def. 5.25 in [3])  
needed for Theorem 4.5, Definition 5.8, Theorem 6.2,
- epi -  $\mathcal{M}$  factorization needed for Lemma 5.2,
- $\mathcal{M} - \mathcal{M}'$  PO-PB decomposition property (see Def. 5.27 in [3])  
needed for Theorem 4.4, Definition 5.8, Theorem 6.2,
- $\mathcal{M} - \mathcal{Q}$  PO-PB decomposition property (see Def. 5.25 in [3])  
needed for Lemma 2.11,
- initial PO over  $\mathcal{M}'$  - morphisms (see Def. 6.1 in [3])  
needed for Theorem 6.3,
- $\mathcal{M}'$  is closed under PO's and PB's along  $\mathcal{M}$  - morphisms  
needed for Theorem 4.5, Definition 5.8, Theorem 6.3,
- $\mathcal{Q}$  is closed under PO's and PB's along  $\mathcal{M}$  - morphisms  
needed for Lemma 2.11, Lemma 5.2,
- induced PB-PO property for  $\mathcal{M}$  and  $\mathcal{Q}$  (see Def. 2.4)  
needed for Lemma 5.2,
- If  $f : A \rightarrow B \in \mathcal{Q}$  and  $g : B \rightarrow C \in \mathcal{M}'$  then  $g \circ f \in \mathcal{Q}$ .  
Composition property for morphisms in  $\mathcal{M}'$  and  $\mathcal{Q}$ ,  
needed for Theorem 4.4,
- If  $g \circ f \in \mathcal{Q}$  and  $g \in \mathcal{M}'$  then  $f \in \mathcal{Q}$ .  
Decomposition property for morphisms in  $\mathcal{M}'$  and  $\mathcal{Q}$ ,  
needed for Theorem 4.5,
- $\mathcal{Q}$  is closed under composition and decomposition  
needed for Lemma 5.2, Lemma 2.11, Theorem 6.6.

Note that these properties hold in particular for the case of graph transformation systems with NACs with  $\mathcal{Q} = \mathcal{M}' = \mathcal{M}$ , where  $\mathcal{M}$  is the set of all graph monomorphisms and  $\mathcal{E}'$  the set of jointly surjective pairs of graph morphisms.

**Definition 2.4** [induced PB-PO property for  $\mathcal{M}$  and  $\mathcal{Q}$ ] Given  $a : A \rightarrow C \in \mathcal{Q}$  and  $b : B \rightarrow C \in \mathcal{M}$  and the following PB and PO

$$\begin{array}{ccc} D & \xrightarrow{d_2} & B \\ d_1 \downarrow & (PB) & \downarrow b \\ A & \xrightarrow{a} & C \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{d_2} & B \\ d_1 \downarrow & (PO) & \downarrow e_1 \\ A & \xrightarrow{e_2} & E \end{array}$$

then the induced morphism  $x : E \rightarrow C$  with  $x \circ e_1 = b$  and  $x \circ e_2 = a$  is a monomorphism in  $\mathcal{Q}$ .

**Remark 2.5** Theorem 5.1 in [8] proves this property in adhesive categories for  $a, b$  being mono with the result that  $x$  is also mono.

A negative application condition or NAC as introduced in [4] forbids a certain structure to be present before or after applying the rule.

**Definition 2.6** [negative application condition, rule with NACs]

- A *negative application condition* or  $NAC(n)$  on  $L$  is an arbitrary morphism  $n : L \rightarrow N$ . A morphism  $g : L \rightarrow G$  satisfies  $NAC(n)$  on  $L$  i.e.  $g \models NAC(n)$  if and only if  $\exists q : N \rightarrow G \in \mathcal{Q}$  such that  $q \circ n = g$ .

$$\begin{array}{ccc} L & \xrightarrow{n} & N \\ \downarrow g & & \searrow \text{---} X_q \\ G & \xleftarrow{\text{---}} & \end{array}$$

A set of NACs on  $L$  is denoted by  $NAC_L = \{NAC(n_i) | i \in I\}$ . A morphism  $g : L \rightarrow G$  satisfies  $NAC_L$  if and only if  $g$  satisfies all single NACs on  $L$  i.e.  $g \models NAC(n_i) \forall i \in I$ .

- A set of NACs  $NAC_L$  (resp.  $NAC_R$ ) on  $L$  (resp.  $R$ ) for a rule  $p : L \xleftarrow{l} K \xrightarrow{r} R$  (with  $l, r \in \mathcal{M}$ ) is called *left* (resp. *right*) NAC on  $p$ .  $NAC_p = (NAC_L, NAC_R)$ , consisting of a set of left and a set of right NACs on  $p$  is called a *set of NACs on  $p$* . A rule  $(p, NAC_p)$  with NACs is a rule with a set of NACs on  $p$ .

**Definition 2.7** [adhesive HLR system with NACs]

- An *adhesive HLR system with NACs*  $AHS = (\mathbf{C}, \mathcal{M}, \mathcal{M}', \mathcal{E}', \mathcal{Q}, P)$  consists of an adhesive HLR category with NACs  $(\mathbf{C}, \mathcal{M}, \mathcal{M}', \mathcal{E}', \mathcal{Q})$  and a set of rules with NACs  $P$ .
- A *direct transformation*  $G \xrightarrow{p, g} H$  via a rule  $p : L \leftarrow K \rightarrow R$  with  $NAC_p = (NAC_L, NAC_R)$  and a match  $g : L \rightarrow G$  consists of the double pushout [2] (DPO)

$$\begin{array}{ccccc} L & \longleftarrow & K & \longrightarrow & R \\ \downarrow g & & \downarrow & & \downarrow h \\ G & \longleftarrow & D & \longrightarrow & H \end{array}$$

where  $g$  satisfies  $NAC_L$ , written  $g \models NAC_L$  and  $h : R \rightarrow H$  satisfies  $NAC_R$ , written  $h \models NAC_R$ . Since pushouts along  $\mathcal{M}$ -morphisms in an adhesive HLR category always exist, the DPO can be constructed if the pushout complement of  $K \rightarrow L \rightarrow G$  exists. If so, we say that the match  $g$  satisfies the gluing condition of rule  $p$ . A *transformation*, denoted as  $G_0 \xRightarrow{*} G_n$ , is a sequence  $G_0 \Rightarrow G_1 \Rightarrow \dots \Rightarrow G_n$  of direct transformations.

**Remark 2.8** From now on we consider only adhesive HLR systems with rules having an empty set of right negative application conditions. This is without loss of generality, because each right NAC can be translated into an equivalent left NAC as explained in [3], where Definition 7.16 and Theorem 7.17 can be specialized to NACs as shown in the following construction and lemma.

**Definition 2.9** [construction of left from right NACs] For each  $NAC(n_i)$  on  $R$  with  $n_i : R \rightarrow N_i$  of a rule  $p = (L \leftarrow K \rightarrow R)$ , the equivalent left application

condition  $L_p(NAC(n_i))$  is defined in the following way:

$$\begin{array}{ccccc} L & \longleftarrow & K & \longrightarrow & R \\ n'_i \downarrow & (2) & \downarrow & (1) & \downarrow n_i \\ N'_i & \longleftarrow & Z & \longrightarrow & N_i \end{array}$$

- If the pair  $(K \rightarrow R, R \rightarrow N_i)$  has a pushout complement, we construct  $(K \rightarrow Z, Z \rightarrow N_i)$  as the pushout complement (1). Then we construct pushout (2) with the morphism  $n'_i : L \rightarrow N'_i$ . Now we define  $L_p(NAC(n_i)) = NAC(n'_i)$ .
- If the pair  $(K \rightarrow R, R \rightarrow N_i)$  does not have a pushout complement, we define  $L_p(NAC(n_i)) = \text{true}$ .

For each set of NACs on  $R$ ,  $NAC_R = \cup_{i \in I} NAC(n_i)$  we define the following set of left NACs:

$$L_p(NAC_R) = \cup_{i \in I'} L_p(NAC(n'_i))$$

with  $i \in I'$  if and only if the pair  $(K \rightarrow R, R \rightarrow N_i)$  has a pushout complement.

**Remark 2.10** Note that  $Z$  is unique since pushout complements along  $\mathcal{M}$ -morphisms are unique up to isomorphism in adhesive HLR categories.

**Lemma 2.11 (equivalence of left and right NACs)** *For every rule  $p$  with  $NAC_R$  a set of right NACs on  $p$ ,  $L_p(NAC_R)$  as defined in Definition 2.9 is a set of left NACs on  $p$  such that for all direct transformations  $G \xrightarrow{p,g} H$  with comatch  $h$ ,*

$$g \models L_p(NAC_R) \Leftrightarrow h \models NAC_R$$

**Proof** The proof corresponds to case 1 and 3 in the proof of Theorem 7.17 in [3].  $\square$

**Definition 2.12** [inverse rule with NACs] For a rule  $p : L \leftarrow K \rightarrow R$  with  $NAC_p = (NAC_L, \emptyset)$ , the inverse rule is defined by  $p^{-1} = R \leftarrow K \rightarrow L$  with  $NAC_{p^{-1}} = (L_{p^{-1}}(NAC_L), \emptyset)$ .

**Theorem 2.13 (Inverse Direct Transformation with NACs)** *For each direct transformation with NACs  $G \Rightarrow H$  via a rule  $p : L \leftarrow K \rightarrow R$  with  $NAC_p$  a set of left NACs on  $p$ , there exists an inverse direct transformation with NACs  $H \Rightarrow G$  via the inverse rule  $p^{-1}$  with  $NAC_{p^{-1}}$ .*

**Proof** This follows directly from Def. 2.12 and Lemma 2.11.  $\square$

### 3 Parallelism in Adhesive HLR Systems with NACs

In order to generalize the notion of parallelism to adhesive HLR systems with NACs at first it is necessary to define when two direct transformations with NACs are parallel independent. For a pair of transformations with NACs it is not only possible that one transformation deletes a structure which is needed by the other one, but also that one transformation produces a structure which is forbidden by the other one. For this new notion of parallel independence it is possible to formulate the local Church-Rosser property with NACs and also a Parallelism Theorem with NACs as described in this section.



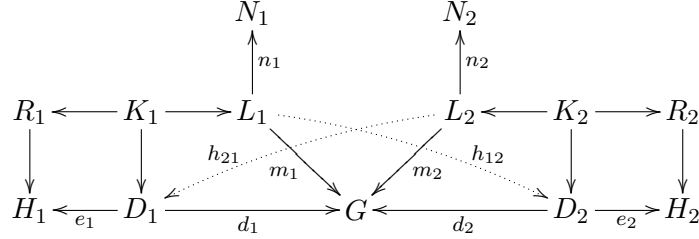
**Definition 3.1** [parallel and sequential independence] Two direct transformations  $G \xRightarrow{(p_1, m_1)} H_1$  with  $NAC_{p_1}$  and  $G \xRightarrow{(p_2, m_2)} H_2$  with  $NAC_{p_2}$  are *parallel independent* if

$$\exists h_{12} : L_1 \rightarrow D_2 \text{ s.t. } (d_2 \circ h_{12} = m_1 \text{ and } e_2 \circ h_{12} \models NAC_{p_1})$$

and

$$\exists h_{21} : L_2 \rightarrow D_1 \text{ s.t. } (d_1 \circ h_{21} = m_2 \text{ and } e_1 \circ h_{21} \models NAC_{p_2})$$

as in the following diagram:



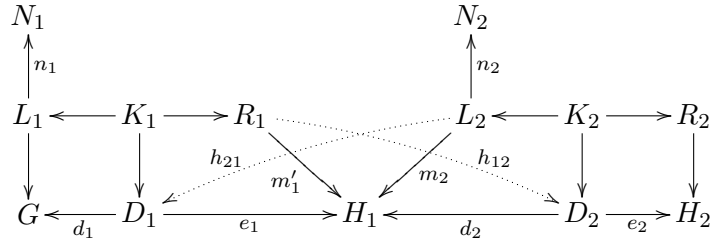
Two direct transformations  $G \xRightarrow{(p_1, m_1)} H_1$  with  $NAC_{p_1}$  and  $H_1 \xRightarrow{(p_2, m_2)} H_2$  with  $NAC_{p_2}$  are *sequentially independent* if

$$\exists h_{12} : R_1 \rightarrow D_2 \text{ s.t. } (d_2 \circ h_{12} = m'_1 \text{ and } e_2 \circ h_{12} \models NAC_{p_1^{-1}})$$

and

$$\exists h_{21} : L_2 \rightarrow D_1 \text{ s.t. } (e_1 \circ h_{21} = m_2 \text{ and } d_1 \circ h_{21} \models NAC_{p_2})$$

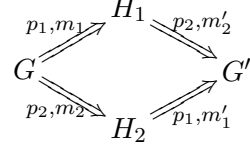
as in the following diagram:



**Remark 3.2** Note that as for the case without NACs we have the following relationship between parallel and sequential independency:  $G \xRightarrow{p_1} H_1 \xRightarrow{p_2} H_2$  are sequentially independent iff  $G \xRightarrow{p_1^{-1}} H_1 \xRightarrow{p_2} H_2$  are parallel independent.

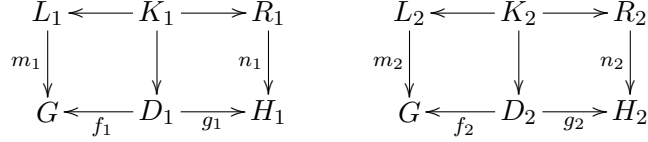
**Theorem 3.3 (Local Church-Rosser Theorem with NACs)** *Given an adhesive HLR system with NACs AHS and two parallel independent direct transformations with NACs  $H_1 \xRightarrow{p_1, m_1} G \xRightarrow{p_2, m_2} H_2$ , there are an object  $G'$  and direct transformations  $H_1 \xRightarrow{p_2, m'_2} G'$  and  $H_2 \xRightarrow{p_1, m'_1} G'$  such that  $G \xRightarrow{p_1, m_1} H_1 \xRightarrow{p_2, m'_2} G'$  and  $G \xRightarrow{p_2, m_2} H_2 \xRightarrow{p_1, m'_1} G'$  are sequentially independent. Vice versa, given two sequentially independent direct transformations with NACs  $G \xRightarrow{p_1, m_1} H_1 \xRightarrow{p_2, m'_2} G'$  there are an object  $H_2$  and sequentially independent direct transformations  $G \xRightarrow{p_2, m_2} H_2 \xRightarrow{p_1, m'_1} G'$*

such that  $H_1 \xleftarrow{p_1, m_1} G \xrightarrow{p_2, m_2} H_2$  are parallel independent:

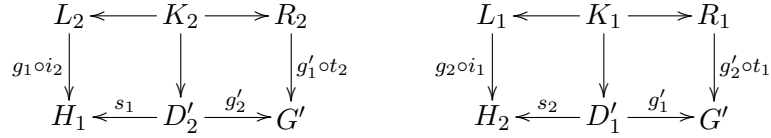


### Proof

(i) Given the parallel independent transformations  $H_1 \xleftarrow{p_1, m_1} G \xrightarrow{p_2, m_2} H_2$  :



Because of Def. 3.1 and the parallel independence with NACs of  $H_1 \xleftarrow{p_1, m_1} G \xrightarrow{p_2, m_2} H_2$  we know that there exists  $i_2 : L_2 \rightarrow D_1$  (resp.  $i_1 : L_1 \rightarrow D_2$ ) s.t.  $f_1 \circ i_2 = m_2$  (resp.  $f_2 \circ i_1 = m_1$ ) and moreover  $g_1 \circ i_2 \models NAC_{p_2}$  (resp.  $g_2 \circ i_1 \models NAC_{p_1}$ ). Because of the Local Church-Rosser Theorem for parallel independent transformations without NACs all necessary pushouts in  $H_1 \xrightarrow{p_2} G'$  and  $H_2 \xrightarrow{p_1} G'$  can be constructed s.t.  $G \xrightarrow{p_1} H_1 \xrightarrow{p_2} G'$  and  $G \xrightarrow{p_2} H_2 \xrightarrow{p_1} G'$  are sequentially independent according to Def. 5.9 in [3] for direct transformations without NACs. This means in particular that  $t_1 : R_1 \rightarrow D'_2$  (resp.  $t_2 : R_2 \rightarrow D'_1$ ) exist s.t.  $s_1 \circ t_1 = n_1$  (resp.  $s_2 \circ t_2 = n_2$ ) and the following pushout diagrams exist:



Since  $g_1 \circ i_2 \models NAC_{p_2}$  and  $g_2 \circ i_1 \models NAC_{p_1}$ ,  $H_1 \xrightarrow{p_2} G'$  and  $H_2 \xrightarrow{p_1} G'$  are valid direct transformations with NACs. For the sequential independence of  $G \xrightarrow{p_1} H_1 \xrightarrow{p_2} G'$  we have to show that  $i_2, t_1$  are the required morphisms. For  $i_2$  we have  $f_1 \circ i_2 = m_2$ , and therefore  $f_1 \circ i_2 \models NAC_{p_2}$  follows by assumption. Now we investigate  $g_2' \circ t_1$ . Because of Theorem 2.13 and the fact that  $g_2 \circ i_1 \models NAC_{p_1}$  it follows directly that also  $g_2' \circ t_1 \models NAC_{p_1^{-1}}$ . Analogously the sequential independence of  $G \xrightarrow{p_2} H_2 \xrightarrow{p_1} G'$  can be proven.

(ii) Given sequentially independent direct transformations with NACs  $G \xleftarrow{p_1, m_1} H_1 \xrightarrow{p_2, m_2'} G'$  with comatches  $n_1'$  and  $n_2'$ , respectively, from Remark 3.2 we obtain parallel independent direct transformations with NACs  $G \xleftarrow{p_1^{-1}, n_1'} H_1 \xrightarrow{p_2, m_2'} G'$ . Now part (i) of the proof gives us sequentially independent direct transformations with NACs  $H_1 \xleftarrow{p_1^{-1}, n_1'} G \xrightarrow{p_2, m_2} H_2$  and  $H_1 \xrightarrow{p_2, m_2'} G' \xleftarrow{p_1^{-1}, n_1'} H_2$ . Applying again Remark 3.2 to the first transformation we obtain parallel independent

direct transformations with NACs  $H_1 \xRightarrow{p_1, m_1} G \xRightarrow{p_2, m_2} H_2$ :

$$\begin{array}{ccccc} & & G & & \\ & \nearrow^{p_1^{-1}, n_1} & & \searrow_{p_2, m_2} & \\ H_1 & & & & H_2 \\ & \searrow_{p_2, m'_2} & & \nearrow_{p_1^{-1}, n'_1} & \\ & & G' & & \end{array}$$

□

**Definition 3.4** [parallel rule with NAC] Let  $AHS = (\mathbf{C}, \mathcal{M}, \mathcal{M}', \mathcal{E}', \mathcal{Q}, P)$  be an adhesive HLR system with NACs, where  $(\mathbf{C}, \mathcal{M}, \mathcal{Q})$  has binary coproducts compatible with  $\mathcal{M}$ . Given two rules  $p_1 = (L_1 \xleftarrow{l_1} K_1 \xrightarrow{r_1} R_1)$  with  $NAC_{p_1}$  and  $p_2 = (L_2 \xleftarrow{l_2} K_2 \xrightarrow{r_2} R_2)$  with  $NAC_{p_2}$ , the *parallel rule*  $p_1 + p_2$  with  $NAC_{p_1+p_2}$  is defined by the coproduct constructions over the corresponding objects and morphisms:  $p_1 + p_2 = (L_1 + L_2 \xleftarrow{l_1+l_2} K_1 + K_2 \xrightarrow{r_1+r_2} R_1 + R_2)$  and  $NAC_{p_1+p_2} = \{n_1 + id_{L_2} | n_1 \in NAC_{p_1}\} \cup \{id_{L_1} + n_2 | n_2 \in NAC_{p_2}\}$ .

**Remark 3.5** It is not sufficient as in the case without NACs to define a direct parallel transformation simply as the application of a parallel rule  $p_1 + p_2$ . For the case with NACs the application of a parallel rule with NACs is a direct parallel transformation with NACs if an extra condition, called NAC-compatibility, is satisfied. This condition expresses that the NACs on rule  $p_1$  and  $p_2$  are satisfied by all matches occurring in the direct transformations when sequentializing the direct parallel transformation.

**Definition 3.6** [parallel transformation with NAC] Given a parallel rule  $p_1 + p_2$  with  $NAC_{p_1+p_2}$ . Consider a direct transformation  $G \Rightarrow G'$  via  $p_1 + p_2$  with  $NAC_{p_1+p_2}$  and a match  $m : L_1 + L_2 \rightarrow G$ . This transformation is *NAC-compatible* if and only if for each  $n_1 \in NAC_{p_1}$  (resp.  $n_2 \in NAC_{p_2}$ )  $\nexists q_1, q'_1 \in \mathcal{Q}$  (resp.  $\nexists q_2, q'_2 \in \mathcal{Q}$ ) s.t.

- $(q_1 + m_2) \circ (n_1 + id_{L_2}) = m_1 + m_2$  (resp.  $(m_1 + q_2) \circ (id_{L_1} + n_2) = m_1 + m_2$ ) and
- $(q'_1 + m'_2) \circ (n_1 + id_{L_2}) = m'_1 + m'_2$  (resp.  $(m'_1 + q'_2) \circ (id_{L_1} + n_2) = m'_1 + m'_2$ )

$$\begin{array}{ccc} N_1 + L_2 \xleftarrow{n_1 + id_{L_2}} L_1 + L_2 & & N_1 + L_2 \xleftarrow{n_1 + id_{L_2}} L_1 + L_2 \\ \searrow_{q_1 + m_2} \downarrow_{m_1 + m_2} & & \searrow_{q'_1 + m'_2} \downarrow_{m'_1 + m'_2} \\ G + G & & H_1 + H_2 \\ \\ L_1 + N_2 \xleftarrow{id_{L_1} + n_2} L_1 + L_2 & & L_1 + N_2 \xleftarrow{id_{L_1} + n_2} L_1 + L_2 \\ \searrow_{m_1 + q_2} \downarrow_{m_1 + m_2} & & \searrow_{m'_1 + q'_2} \downarrow_{m'_1 + m'_2} \\ G + G & & H_1 + H_2 \end{array}$$

with  $m_1, m_2$  the matches of the direct transformations  $G \Rightarrow H_1$  and  $G \Rightarrow H_2$  via  $p_1$  resp.  $p_2$  and  $m'_1$  and  $m'_2$  the matches of the direct transformations  $H_2 \Rightarrow G'$  and  $H_1 \Rightarrow G'$  via  $p_1$  resp.  $p_2$  as constructed in the Parallelism Theorem without NACs (Analysis part in Theorem 5.18 in [3]). A NAC-compatible direct parallel transformation via  $p_1 + p_2$  and  $m$  is called a *direct parallel transformation with NACs* or *parallel transformation with NACs*, for short.

**Theorem 3.7 (Parallelism Theorem with NACs)** Let  $AHS = (\mathbf{C}, \mathcal{M}, \mathcal{M}', \mathcal{E}', \mathcal{Q}, P)$  be an adhesive HLR system with NACs, where  $(\mathbf{C}, \mathcal{M})$

has binary coproducts compatible with  $\mathcal{M}$ .

- (i) *Synthesis.* Given a sequentially independent direct transformation sequence with NACs  $G \Rightarrow H_1 \Rightarrow G'$  via  $p_1, m_1$  (resp.  $p_2, m'_2$ ) with  $NAC_{p_1}$  (resp.  $NAC_{p_2}$ ), then there is a construction leading to a parallel transformation with NACs  $G \Rightarrow G'$  via  $[m_1, m_2]$  and the parallel rule  $p_1 + p_2$  with  $NAC_{p_1+p_2}$ , called a synthesis construction.
- (ii) *Analysis.* Given a direct parallel transformation with NACs  $G \Rightarrow G'$  via  $m : L_1 + L_2 \rightarrow G$  and the parallel rule  $p_1 + p_2$  with  $NAC_{p_1+p_2}$ , then there is a construction leading to two sequentially independent transformation sequences with NACs  $G \Rightarrow H_1 \Rightarrow G'$  via  $p_1, m_1$  and  $p_2, m'_2$  and  $G \Rightarrow H_2 \Rightarrow G'$  via  $p_2, m_2$  and  $p_1, m'_1$ , called an analysis construction.
- (iii) *Bijective Correspondence.* The synthesis and analysis constructions are inverse to each other up to isomorphism.

$$\begin{array}{ccccc}
 & & G & & \\
 & \swarrow^{p_1, m_1} & & \searrow_{p_2, m_2} & \\
 H_1 & & & & H_2 \\
 & \searrow_{p_2, m'_2} & & \swarrow_{p_1, m'_1} & \\
 & & G' & & 
 \end{array}$$

### Proof

- (i) Given the sequentially independent direct transformations with NACs  $G \xRightarrow{p_1, m_1} H_1 \xRightarrow{p_2, m'_2} G'$ , using the Parallelism Theorem (Theorem 5.18 in [3]) without NACs we can construct the following double pushout:

$$\begin{array}{ccccc}
 L_1 + L_2 & \longleftarrow & K_1 + K_2 & \longrightarrow & R_1 + R_2 \\
 \downarrow [m_1, m_2] & & \downarrow & & \downarrow \\
 G & \longleftarrow & D & \longrightarrow & G'
 \end{array}$$

Now we have to prove that this direct transformation is NAC-compatible. Note at first that because of Theorem 3.3 we have also the following two sequentially independent direct transformations with NACs  $G \Rightarrow H_2 \Rightarrow G'$  via  $p_2, m_2$  and  $p_1, m'_1$ . We now show that for each  $n_1 \in NAC_{p_1}$  (resp.  $n_2 \in NAC_{p_2}$ )  $\nexists q_1, q'_1 \in \mathcal{Q}$  (resp.  $\nexists q_2, q'_2 \in \mathcal{Q}$ ) s.t.

- $(q_1 + m_2) \circ (n_1 + id_{L_2}) = m_1 + m_2$  (resp.  $(m_1 + q_2) \circ (id_{L_1} + n_2) = m_1 + m_2$ ) and
  - $(q'_1 + m'_2) \circ (n_1 + id_{L_2}) = m'_1 + m'_2$  (resp.  $(m'_1 + q'_2) \circ (id_{L_1} + n_2) = m'_1 + m'_2$ )
- Suppose that  $q_1$  would exist satisfying the first property. Then it follows in particular that  $q_1 \circ n_1 = m_1$  and this means that  $m_1 \not\models NAC_{p_1}$  which is a contradiction. Suppose moreover that  $q'_1$  exists satisfying the second property. Then it follows that  $q'_1 \circ n_1 = m'_1$  and this would mean that  $m'_1 \not\models NAC_{p_1}$  which is a contradiction as well. We can argue analogously for  $q_2$  and  $q'_2$ .

- (ii) Given a parallel transformation with NACs  $G \xRightarrow{p_1+p_2, m} G'$  then because of the Parallelism Theorem (Theorem 5.18 in [3]) without NACs it follows that  $G \Rightarrow H_1$  and  $G \Rightarrow H_2$  are parallel independent without NACs and moreover the necessary double pushouts for  $G \Rightarrow H_1 \Rightarrow G'$  via  $p_1, m_1$  and  $p_2, m'_2$  and

$G \Rightarrow H_2 \Rightarrow G'$  via  $p_2, m_2$  and  $p_1, m'_1$  can be constructed s.t. they are sequentially independent without NACs. If we can prove that  $m_1, m'_1 \models NAC_{p_1}$  and  $m_2, m'_2 \models NAC_{p_2}$ , then we know that  $G \Rightarrow H_1$  and  $G \Rightarrow H_2$  are parallel independent as transformations with NACs. We know that the direct transformation  $G \xrightarrow{p_1+p_2, m} G'$  is NAC-compatible. So we use the fact that for each  $n_1 \in NAC_{p_1}$  (resp.  $n_2 \in NAC_{p_2}$ )  $\nexists q_1, q'_1 \in \mathcal{Q}$  (resp.  $\nexists q_2, q'_2 \in \mathcal{Q}$ ) s.t.

- $(q_1 + m_2) \circ (n_1 + id_{L_2}) = m_1 + m_2$  (resp.  $(m_1 + q_2) \circ (id_{L_1} + n_2) = m_1 + m_2$ ) and
  - $(q'_1 + m'_2) \circ (n_1 + id_{L_2}) = m'_1 + m'_2$  (resp.  $(m'_1 + q'_2) \circ (id_{L_1} + n_2) = m'_1 + m'_2$ )
- Suppose that  $m_1 \not\models NAC_{p_1}$ . Then there exists some  $q_1 \in \mathcal{Q}$  s.t.  $q_1 \circ n_1 = m_1$  with  $n_1 \in NAC_{p_1}$ . Then the first property for  $q_1$  would hold and this is a contradiction. Suppose that  $m'_2 \not\models NAC_{p_2}$ . Then there exists some  $q'_2 \in \mathcal{Q}$  s.t.  $q'_2 \circ n_2 = m'_2$  with  $n_2 \in NAC_{p_2}$ . Then the second property for  $q'_2$  would hold and this is a contradiction. We can argue analogously for  $m_2$  and  $m'_1$ .

Now we have proven that  $G \Rightarrow H_1$  and  $G \Rightarrow H_2$  are parallel independent with NACs and from Theorem 3.3 it follows that  $G \Rightarrow H_1 \Rightarrow G'$  and  $G \Rightarrow H_2 \Rightarrow G'$  are then sequentially independent with NACs.

- (iii) Because of the uniqueness of pushouts and pushout complements, the constructions are inverse to each other up to isomorphism.  $\square$

The following lemma allows an elegant characterization of conflicts (Def. 3.9) in Lemma 3.10.

**Lemma 3.8 (Unique Match)** *Given two direct transformations  $G \xrightarrow{(p_1, m_1)} H_1$  with  $NAC_{p_1}$  and  $G \xrightarrow{(p_2, m_2)} H_2$  with  $NAC_{p_2}$ , then the following holds:*

- if  $\exists h_{12} : L_1 \rightarrow D_2$  s.t.  $d_2 \circ h_{12} = m_1$  then  $h_{12}$  is unique,
- if  $\exists h_{21} : L_2 \rightarrow D_1$  s.t.  $d_1 \circ h_{21} = m_2$  then  $h_{21}$  is unique.

**Proof** Since each rule consists of two morphisms in  $\mathcal{M}$  and  $\mathcal{M}$ -morphisms are closed under pushouts  $d_1$  and  $d_2$  are in  $\mathcal{M}$  as well. Suppose there exists  $h'_{12} : L_1 \rightarrow D_2 : d_2 \circ h'_{12} = m_1$  then because of  $d_2$  in  $\mathcal{M}$  and thus a monomorphism and  $d_2 \circ h'_{12} = d_2 \circ h_{12} = m_1$  it follows that  $h'_{12} = h_{12}$ . Analogously one can prove that  $h_{21}$  is unique.  $\square$

**Definition 3.9** [conflict] Two direct transformations  $G \xrightarrow{(p_1, m_1)} H_1$  with  $NAC_{p_1}$  and  $G \xrightarrow{(p_2, m_2)} H_2$  with  $NAC_{p_2}$  are *in conflict* if they are not parallel independent i.e. if

$$\nexists h_{12} : L_1 \rightarrow D_2 \text{ s.t. } (d_2 \circ h_{12} = m_1 \text{ and } e_2 \circ h_{12} \models NAC_{p_1})$$

or

$$\nexists h_{21} : L_2 \rightarrow D_1 \text{ s.t. } (d_1 \circ h_{21} = m_2 \text{ and } e_1 \circ h_{21} \models NAC_{p_2}).$$

In the following conflict characterization it is described which type of conflicts may arise between a pair of direct transformations  $G \xrightarrow{(p_1, m_1)} H_1$  and  $G \xrightarrow{(p_2, m_2)} H_2$ . The so-called use-delete-conflict occurs when rule  $p_1$  applied to  $H_2$  wants to use some item which is deleted by  $p_2$  applied to  $G$  leading to  $H_2$ . A forbid-produce-conflict

occurs when rule  $p_1$  applied to  $H_2$  forbids some item according to  $NAC_{p_1}$  which is produced by  $p_2$ . Analogously we can have a delete-use- or produce-forbid-conflict.

**Lemma 3.10 (Conflict Characterization)** *Two direct transformations  $G \xrightarrow{(p_1, m_1)} H_1$  with  $NAC_{p_1}$  and  $G \xrightarrow{(p_2, m_2)} H_2$  with  $NAC_{p_2}$  are in conflict if and only if:*

- (i) (a)  $\nexists h_{12} : L_1 \rightarrow D_2 : d_2 \circ h_{12} = m_1$  (use-delete-conflict)  
or  
(b) *there exists a unique  $h_{12} : L_1 \rightarrow D_2 : d_2 \circ h_{12} = m_1$ , but  $e_2 \circ h_{12} \not\models NAC_{p_1}$  (forbid-produce-conflict)*  
or
- (ii) (a)  $\nexists h_{21} : L_2 \rightarrow D_1 : d_1 \circ h_{21} = m_2$  (delete-use-conflict)  
or  
(b) *there exists a unique  $h_{21} : L_2 \rightarrow D_1 : d_1 \circ h_{21} = m_2$ , but  $e_1 \circ h_{21} \not\models NAC_{p_2}$  (produce-forbid-conflict).*

**Proof**  $G \xrightarrow{(p_1, m_1)} H_1$  with  $NAC_{p_1}$  and  $G \xrightarrow{(p_2, m_2)} H_2$  with  $NAC_{p_2}$  are in conflict if

$$\nexists h_{12} : L_1 \rightarrow D_2 \text{ s.t. } (d_2 \circ h_{12} = m_1 \text{ and } e_2 \circ h_{12} \models NAC_{p_1})$$

or

$$\nexists h_{21} : L_2 \rightarrow D_1 \text{ s.t. } (d_1 \circ h_{21} = m_2 \text{ and } e_1 \circ h_{21} \models NAC_{p_2})$$

We consider at first the first line of this disjunction. Let  $A(h_{12}) := d_2 \circ h_{12} = m_1$ ,  $B(h_{12}) := e_2 \circ h_{12} \models NAC_{p_1}$ ,  $P(h_{12}) := (A(h_{12}) \wedge B(h_{12}))$  and  $\mathcal{M}_{12}$  be the set of all morphisms from  $L_1$  to  $D_2$ . Then the first line is equivalent to

$$\nexists h_{12} \in \mathcal{M}_{12} : (A(h_{12}) \wedge B(h_{12})) \equiv \nexists h_{12} \in \mathcal{M}_{12} : P(h_{12})$$

This is equivalent to

$$\forall h_{12} \in \mathcal{M}_{12} : \neg P(h_{12}) \equiv (\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg P(h_{12}))$$

Moreover  $P \equiv A \wedge B \equiv A \wedge (A \Rightarrow B)$  and thus  $\neg P \equiv \neg(A \wedge B) \equiv \neg(A \wedge (A \Rightarrow B)) \equiv \neg A \vee \neg(A \Rightarrow B) \equiv \neg A \vee \neg(\neg A \vee B) \equiv \neg A \vee (A \wedge \neg B)$ . This implies that  $(\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg P(h_{12})) \equiv$

$$(\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg A(h_{12}) \vee (A(h_{12}) \wedge \neg B(h_{12})))$$

Because of Lemma 3.8 and because the disjunction holding for each morphism in  $\mathcal{M}_{12}$  is an exclusive one this is equivalent to

$$(\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg A(h_{12})) \vee (\exists! h_{12} \in \mathcal{M}_{12} : (A(h_{12}) \wedge \neg B(h_{12})))$$

Now  $(\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg A(h_{12})) \equiv \forall h_{12} \in \mathcal{M}_{12} : \neg A(h_{12}) \equiv \nexists h_{12} \in \mathcal{M}_{12} : A(h_{12})$ . This implies finally that  $\nexists h_{12} : L_1 \rightarrow D_2 \text{ s.t. } (d_2 \circ h_{12} = m_1 \text{ and } e_2 \circ h_{12} \models NAC_{p_1})$  is equivalent to

$$(\nexists h_{12} \in \mathcal{M}_{12} : d_2 \circ h_{12} = m_1) \vee (\exists! h_{12} \in \mathcal{M}_{12} : (d_2 \circ h_{12} = m_1 \wedge e_2 \circ h_{12} \not\models NAC_{p_1}))$$

is equivalent to

- (i) (a)  $\nexists h_{12} : L_1 \rightarrow D_2 : d_2 \circ h_{12} = m_1$  (use-delete-conflict)  
or
- (b) there exists a unique  $h_{12} : L_1 \rightarrow D_2 : d_2 \circ h_{12} = m_1$ , but  $e_2 \circ h_{12} \not\models NAC_{p_1}$  (forbid-produce-conflict)

Analogously we can proceed for the second part of the disjunction.  $\square$

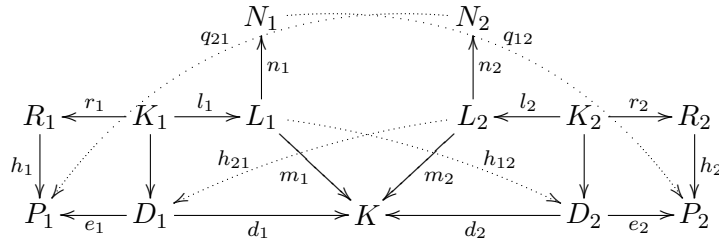
Note that a use-delete-conflict (resp. delete-use-conflict) cannot occur simultaneously to a forbid-produce-conflict (resp. produce-forbid-conflict), since  $(1.a) \Rightarrow \neg(1.b)$  (resp.  $(2.a) \Rightarrow \neg(2.b)$ ). The following types of conflicts can occur simultaneously though: use-delete/delete-use-, use-delete/produce-forbid-, forbid-produce/delete-use-, forbid-produce/produce-forbid-conflicts.

## 4 Critical Pairs for adhesive HLR Systems with NACs

The conflict characterization formulated in the last section leads to a new critical pair notion for transformations with NACs. A critical pair describes a conflict between two rules in a minimal context. Therefore we come to the following critical pair definition with  $\mathcal{E}'$  expressing this minimal context. Moreover, it is proven in this section that this critical pair definition satisfies completeness.

**Definition 4.1** [critical pair] A critical pair is a pair of direct transformations  $K \xrightarrow{(p_1, m_1)} P_1$  with  $NAC_{p_1}$  and  $K \xrightarrow{(p_2, m_2)} P_2$  with  $NAC_{p_2}$  such that:

- (i) (a)  $\nexists h_{12} : L_1 \rightarrow D_2 : d_2 \circ h_{12} = m_1$  and  $(m_1, m_2)$  in  $\mathcal{E}'$  (use-delete-conflict)  
or
- (b) there exists  $h_{12} : L_1 \rightarrow D_2$  s.t.  $d_2 \circ h_{12} = m_1$ , but for one of the NACs  $n_1 : L_1 \rightarrow N_1$  of  $p_1$  there exists a morphism  $q_{12} : N_1 \rightarrow P_2 \in \mathcal{Q}$  s.t.  $q_{12} \circ n_1 = e_2 \circ h_{12}$  and  $(q_{12}, h_2)$  in  $\mathcal{E}'$  (forbid-produce-conflict)  
or
- (ii) (a)  $\nexists h_{21} : L_2 \rightarrow D_1 : d_1 \circ h_{21} = m_2$  and  $(m_1, m_2)$  in  $\mathcal{E}'$  (delete-use-conflict)  
or
- (b) there exists  $h_{21} : L_2 \rightarrow D_1$  s.t.  $d_1 \circ h_{21} = m_2$ , but for one of the NACs  $n_2 : L_2 \rightarrow N_2$  of  $p_2$  there exists a morphism  $q_{21} : N_2 \rightarrow P_1 \in \mathcal{Q}$  s.t.  $q_{21} \circ n_2 = e_1 \circ h_{21}$  and  $(q_{21}, h_1)$  in  $\mathcal{E}'$  (produce-forbid-conflict)



Now we prove that Definition 4.1 of critical pairs leads to completeness. This means, that each occurring conflict in a transformation system with NACs can be

expressed by a critical pair i.e. by the same kind of conflict but in a minimal context. Therefore at first we need the following definition and lemma.

**Definition 4.2** [extension diagram] An *extension diagram* is a diagram (1),

$$\begin{array}{ccc} G_0 & \xRightarrow{t}^* & G_n \\ k_0 \downarrow & (1) & \downarrow k_n \\ G'_0 & \xRightarrow{t'}^* & G'_n \end{array}$$

where,  $k_0 : G_0 \rightarrow G'_0$  is a morphism, called extension morphism, and  $t : G_0 \xRightarrow{*} G_n$  and  $t' : G'_0 \xRightarrow{*} G'_n$  are transformations via the same rules  $(p_0, \dots, p_{n-1})$ , and matches  $(m_0, \dots, m_{n-1})$  and extended matches  $(k_0 \circ m_0, \dots, k_{n-1} \circ m_{n-1})$ , respectively, defined by the following DPO diagrams :

$$p_i : \begin{array}{ccccc} L_i & \xleftarrow{l_i} & K_i & \xrightarrow{r_i} & R_i \\ m_i \downarrow & & j_i \downarrow & & n_i \downarrow \\ G_i & \xleftarrow{f_i} & D_i & \xrightarrow{g_i} & G_{i+1} \\ k_i \downarrow & & d_i \downarrow & & k_{i+1} \downarrow \\ G'_i & \xleftarrow{f'_i} & D'_i & \xrightarrow{g'_i} & G'_{i+1} \end{array}$$

**Remark 4.3** Since  $t$  and  $t'$  are transformations with NACs, the matches  $(m_0, \dots, m_{n-1})$  and extended matches  $(k_0 \circ m_0, \dots, k_{n-1} \circ m_{n-1})$  have to satisfy the NACs of the rules  $(p_0, \dots, p_{n-1})$ .

**Theorem 4.4 (Restriction Theorem with NACs)** *Given a direct transformation  $G \xRightarrow{p} H$  with NACs via the rule  $p : L \xleftarrow{l} K \xrightarrow{r} R$  and match  $m : L \rightarrow G$  and given an object  $K'$  with two morphisms  $L \xrightarrow{m_{lk}} K' \xrightarrow{m_{kg}} G$  s.t.  $m = m_{kg} \circ m_{lk}$ , with  $m_{kg} \in \mathcal{M}'$ , then there exists a direct transformation leading to the following extension diagram:*

$$\begin{array}{ccccc} & & N & & \\ & & \uparrow n & & \\ & & L & \xleftarrow{l} & K \xrightarrow{r} R \\ & \swarrow m_{lk} & \downarrow k & \downarrow k' & \downarrow h \\ & K' & \xleftarrow{d} & D & \xrightarrow{e} P \\ & \swarrow m_{kg} & \downarrow f & \downarrow f' & \downarrow o \\ & G & \xleftarrow{d'} & D' & \xrightarrow{e'} H \end{array}$$

(1) (2) (3) (4)

**Proof** Given  $G \xRightarrow{p} H$  with NAC  $n$  as shown above. Since  $d' \in \mathcal{M}$  we can take the pullback (3) of  $m_{kg}$  and  $d'$ . Since  $m_{kg} \circ m_{lk} \circ l = m \circ l = d' \circ k'$  then there exists a morphism  $k : K \rightarrow D$  with  $k' = f \circ k$  and  $d \circ k = m_{lk} \circ l$  because of the pullback property of (3). Because of the  $\mathcal{M} - \mathcal{M}'$  pushout-pullback-decomposition property [3],  $l \in \mathcal{M}$  and  $m_{kg} \in \mathcal{M}'$ , diagrams (1) and (3) are both pushouts. Now we can construct pushout (2) of  $D \leftarrow K \rightarrow R$  because of  $r \in \mathcal{M}$ . Since



$e' \circ f \circ k = e' \circ k' = h' \circ r$  there exists a morphism  $o : P \rightarrow H$  with  $o \circ h = h'$  and  $o \circ e = e' \circ f$  because of the pushout-property of (2). Because of the pushout-decomposition property also diagram (4) is a pushout.

It remains to show that  $m_{lk}$  satisfies the NACs of  $p$ . Suppose that  $m_{lk}$  doesn't satisfy some  $NAC(n)$  of  $p$ , then there exists a morphism  $q : N \rightarrow K' \in \mathcal{Q}$  s.t.  $q \circ n = m_{lk}$ , but this implies  $m_{kg} \circ q \circ n = m_{kg} \circ m_{lk} = m$  with  $q \in \mathcal{Q}$ ,  $m_{kg} \in \mathcal{M}'$  and because of the composition property  $m_{kg} \circ q \in \mathcal{Q}$ , and this is a contradiction.  $\square$

**Theorem 4.5 (Completeness of Critical Pairs with NACs)** *For each pair of direct transformations  $H_1 \xleftarrow{(p_1, m'_1)} G \xrightarrow{(p_2, m'_2)} H_2$  in conflict there is a critical pair  $P_1 \xleftarrow{p_1} K \xrightarrow{p_2} P_2$  with extension diagrams (1) and (2) and  $m \in \mathcal{M}'$ .*

$$\begin{array}{ccccc} P_1 & \xleftarrow{\quad} & K & \xrightarrow{\quad} & P_2 \\ \downarrow & & \downarrow & & \downarrow \\ H_1 & \xleftarrow{\quad} & G & \xrightarrow{\quad} & H_2 \end{array} \quad \begin{array}{c} (1) \quad m \\ (2) \end{array}$$

**Proof** According to Lemma 3.10 the following reasons are responsible for a pair of direct transformations  $G \xrightarrow{(p_1, m'_1)} H_1$  with  $NAC_{p_1}$  and  $G \xrightarrow{(p_2, m'_2)} H_2$  with  $NAC_{p_2}$  to be in conflict:

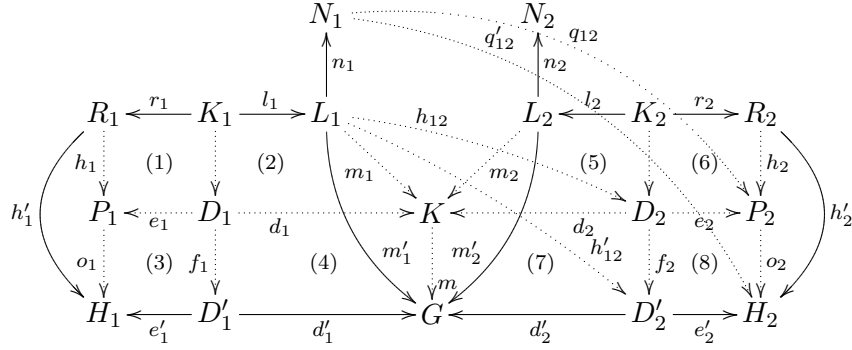
- (i) (a)  $\nexists h'_{12} : L_1 \rightarrow D'_2 : d'_2 \circ h'_{12} = m'_1$  (use-delete-conflict)  
or
- (b) there exists a unique  $h'_{12} : L_1 \rightarrow D'_2 : d'_2 \circ h'_{12} = m'_1$ , but  $e'_2 \circ h'_{12} \not\models NAC_{p_1}$  (forbid-produce-conflict)  
or
- (ii) (a)  $\nexists h'_{21} : L_2 \rightarrow D_1 : d'_1 \circ h'_{21} = m'_2$  (delete-use-conflict)  
or
- (b) there exists a unique  $h'_{21} : L_2 \rightarrow D_1 : d'_1 \circ h'_{21} = m'_2$ , but  $e'_1 \circ h'_{21} \not\models NAC_{p_2}$  (produce-forbid-conflict)

It is possible, that (1.b) and (2.b) are both false. In this case, (1.a) or (2.a) have to be true which corresponds to the usual use-delete-conflict (resp. delete-use-conflict) and in [3] it is described how to embed a critical pair into this pair of direct transformations. In the other case, (1.b) or (2.b) are true. Let at first (1.b) be true. This means that there exists a unique  $h'_{12} : L_1 \rightarrow D'_2 : d'_2 \circ h'_{12} = m'_1$ , but  $e'_2 \circ h'_{12} \not\models NAC_{p_1}$ . Thus for one of the NACs  $n_1 : L_1 \rightarrow N_1$  of  $p_1$  there exists a morphism  $q'_{12} : N_1 \rightarrow H_2 \in \mathcal{Q}$  such that  $q'_{12} \circ n_1 = e'_2 \circ h'_{12}$ . For each pair of morphisms with the same codomain, we have an  $\mathcal{E}' - \mathcal{M}'$  pair factorization. Thus for  $q'_{12} : N_1 \rightarrow H_2$  and  $h'_2 : R_2 \rightarrow H_2$  we obtain an object  $P_2$  and morphisms  $h_2 : R_2 \rightarrow P_2$ ,  $q_{12} : N_1 \rightarrow P_2$  and  $o_2 : P_2 \rightarrow H_2$  with  $(h_2, q_{12}) \in \mathcal{E}'$  and  $o_2 \in \mathcal{M}'$  such that  $o_2 \circ h_2 = h'_2$  and  $o_2 \circ q_{12} = q'_{12}$ . Because of Theorem 4.4 pushouts (5) - (8) can be constructed, if we consider the fact that also  $H_2 \Rightarrow G$  is a direct transformation via  $p_2^{-1}$ . Since  $o_2 \in \mathcal{M}'$  and (7) and (8) are pushouts along  $\mathcal{M}$ -morphisms also  $f_2 \in \mathcal{M}'$  and  $m \in \mathcal{M}'$ . Because of the same argumentation as in Theorem 4.4, since  $m'_2$  satisfies all the NACs of  $p_2$  also  $m_2$  satisfies them. Now we have the first half  $K \Rightarrow P_2$  of the critical pair under construction. Now we can start constructing the second half of the critical pair. Let  $m_1$  be the morphism  $d_2 \circ h_{12}$ , then the following

holds:  $m \circ m_1 = m \circ d_2 \circ h_{12} = d'_2 \circ f_2 \circ h_{12} = d'_2 \circ h'_{12} = m'_1$ . Because of Theorem 4.4 and  $m \in \mathcal{M}'$  pushouts (1) - (4) can be constructed and because of the same argumentation as in Theorem 4.4  $m_1$  satisfies the NACs of  $p_1$ .

We still have to check if this critical pair is in forbid-produce-conflict. Since (8) is a pullback and  $o_2 \circ q_{12} \circ n_1 = q'_{12} \circ n_1 = e'_2 \circ h'_{12}$  there exists a morphism  $h_{12} : L_1 \rightarrow D_2$ , with  $e_2 \circ h_{12} = q_{12} \circ n_1$  and  $f_2 \circ h_{12} = h'_{12}$ . Because of the decomposition property,  $q'_{12} = o_2 \circ q_{12} \in \mathcal{Q}$  and  $o_2 \in \mathcal{M}'$  we have  $q_{12} \in \mathcal{Q}$ . This means that  $e_2 \circ h_{12}$  does not satisfy either the NAC  $n_1 : L_1 \rightarrow N_1$ .

Thus finally we obtain a critical pair according to Def. 4.1 of type (1.b) because we have  $h_{12}$  with  $d_2 \circ h_{12} = m_1$ . Moreover there is  $q_{12} \in \mathcal{Q}$  with  $(q_{12}, h_2) \in \mathcal{E}'$  and  $e_2 \circ h_{12} = q_{12} \circ n_1$ .



We can proceed analogously for the case of (2.b) being true leading to a critical pair of type (2.b) according to Def. 4.1.  $\square$

**Fact 4.6 (necessary and sufficient condition for parallel independence)**

*Each pair of direct transformations  $H_1 \Leftarrow G \Rightarrow H_2$  in an adhesive HLR system with NACs is parallel independent if and only if there are no critical pairs for this adhesive HLR system with NACs. An adhesive HLR system with NACs is locally confluent if there are no critical pairs for this adhesive HLR system with NACs.*

**Proof**

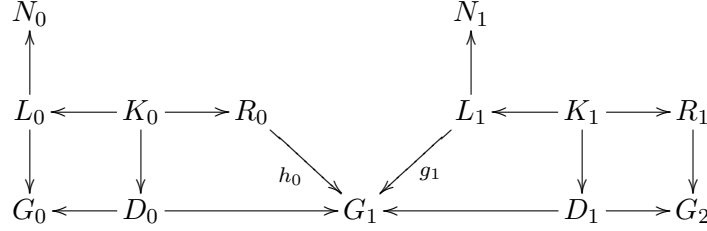
- Given an adhesive HLR system with NACs with an empty set of critical pairs and let  $H_1 \Leftarrow G \Rightarrow H_2$  be a pair of direct transformations in conflict for this adhesive HLR system with NACs. This is a contradiction, since then there would exist a critical pair which can be embedded into this pair of direct transformations as shown in Theorem 4.5.
- Given an adhesive HLR system with NACs with only parallel independent pairs of direct transformations  $H_1 \Leftarrow G \Rightarrow H_2$ . Then the set of critical pairs has to be empty, otherwise a critical pair would be a pair of direct transformations in conflict.
- If each pair of direct transformations  $H_1 \Leftarrow G \Rightarrow H_2$  in an adhesive HLR system with NACs is parallel independent then each pair is also locally confluent and using Theorem 3.3 in consequence this adhesive HLR system with NACs is locally confluent.

$\square$

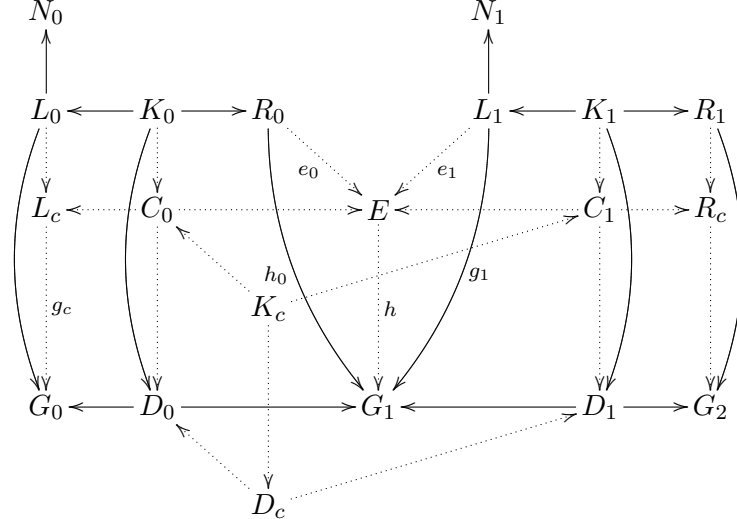
## 5 Concurrency in adhesive HLR Systems with NACs

Let  $t$  be a transformation via the rules  $p_0, \dots, p_{n-1}$  with NACs and matches  $g_0, \dots, g_{n-1}$ . In general there will be dependencies between several direct transformations in this transformation sequence. Therefore it is not possible to apply the Parallelism Theorem in order to summarize the transformation sequence into one equivalent transformation step. It is possible though to formulate a Concurrency Theorem which expresses how to translate such a sequence into one equivalent transformation step anyway. Therefore we build on the notion of a concurrent rule of a transformation sequence without NACs as introduced in [3]. Moreover we have to translate all the NACs occurring in the rule sequence  $p_0, \dots, p_{n-1}$  backward into an equivalent set of NACs on the concurrent rule  $p_c$  of this rule sequence. This means, a set  $NAC_{p_c}$  should be found such that this set of NACs is equivalent to  $NAC_{p_0}, \dots, NAC_{p_{n-1}}$  for the transformation  $t$ . This section describes gradually how to obtain this concurrent NAC and generalizes then the Concurrency Theorem to transformations with NACs.

Let us consider at first a two-step transformation with NACs:



The goal is to translate all NACs on  $p_0$  and  $p_1$  into an equivalent set of NACs on the concurrent rule  $p_c : L_c \leftarrow K_c \rightarrow R_c$  inducing as explained in Theorem 5.23 in [3] a concurrent transformation  $G_0 \Rightarrow G_2$  via  $p_c$  and match  $g_c$  as shown in the following diagram:



Consequently the two necessary steps are:

- Translate each set of NACs on  $L_0$  into an equivalent set of NACs on  $L_c$ .
- Translate each set of NACs on  $L_1$  into an equivalent set of NACs on  $L_c$ .

We can prove the first step as described in the following construction and Lemma.

**Definition 5.1** [construction of NACs on  $L_c$  from NACs on  $L_0$ ] Consider the following diagram:

$$\begin{array}{ccc} N_j & \xrightarrow{e_i} & N'_i \\ n_j \uparrow & (1) & \uparrow n'_i \\ L_0 & \xrightarrow{m_0} & L_c \end{array}$$

For each  $NAC(n_j)$  on  $L_0$  with  $n_j : L_0 \rightarrow N_j$  and  $m_0 : L_0 \rightarrow L_c$ , let

$$D_{m_0}(NAC(n_j)) = \{NAC(n'_i) | i \in I, n'_i : L_c \rightarrow N'_i\}$$

where  $I$  and  $n'_i$  are constructed as follows:  $i \in I$  if and only if  $(e_i, n'_i)$  with  $e_i : N_j \rightarrow N'_i$  jointly epimorphic,  $e_i \circ n_j = n'_i \circ m_0$  and  $e_i \in \mathcal{Q}$ .

For each set of NACs  $NAC_{L_0} = \{NAC(n_j) | j \in J\}$  on  $L_0$  the downward translation of  $NAC_{L_0}$  is then defined as:

$$D_{m_0}(NAC_{L_0}) = \cup_{j \in J} D_{m_0}(NAC(n_j))$$

**Lemma 5.2 (equivalence of set of NACs on  $L_0$  and set of NACs on  $L_c$ )**

Given  $g_c : L_c \rightarrow G_0$ ,  $m_0 : L_0 \rightarrow L_c$  with  $NAC_{L_0}$  and  $g_0 = g_c \circ m_0$  as in the following diagram:

$$\begin{array}{ccc} N_j & \xrightarrow{e_i} & N'_i \\ n_j \uparrow & (1) & \uparrow n'_i \\ L_0 & \xrightarrow{m_0} & L_c \\ \downarrow g_0 & = & \downarrow g_c \\ G_0 & & G_0 \end{array} \quad \begin{array}{c} q \\ \curvearrowright \\ q' \end{array}$$

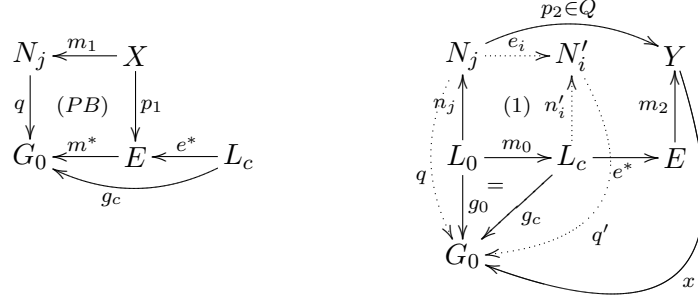
then the following holds :

$$g_0 \models NAC_{L_0} \Leftrightarrow g_c \models D_{m_0}(NAC_{L_0})$$

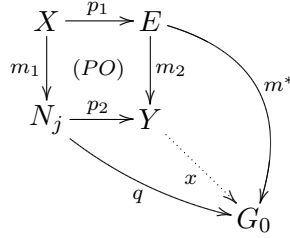
**Proof**

- $(\Rightarrow)$  Let  $g_c \not\models D_{m_0}(NAC(L_0)) = \cup_{j \in J} D_{m_0}(NAC(n_j))$  with  $NAC_{L_0} = \{NAC(n_j) | j \in J\}$ . Then for some  $j \in J$  there is a NAC  $n'_i : L_c \rightarrow N'_i \in D_{m_0}(NAC(n_j))$  and  $e_i : N_j \rightarrow N'_i$  for which holds that  $g_c \not\models NAC(n'_i)$ ,  $(e_i, n'_i)$  jointly epi,  $e_i \in \mathcal{Q}$  and  $e_i \circ n_j = n'_i \circ m_0$ . Consequently there exists a morphism  $q' : N'_i \rightarrow G_0 \in \mathcal{Q}$  such that  $q' \circ n'_i = g_c$ . Since  $g_0 = g_c \circ m_0 = q' \circ n'_i \circ m_0 = q' \circ e_i \circ n_j$  there exists a morphism  $q : N_j \rightarrow G_0$  defined by  $q = q' \circ e_i$  s.t.  $q \circ n_j = q' \circ e_i \circ n_j = g_0$ . Because of the composition property for morphisms in  $\mathcal{Q}$  we have  $q \in \mathcal{Q}$  since  $q' \in \mathcal{Q}$  and  $e_i$  in  $\mathcal{Q}$ . Hence  $g_0 \models NAC(n_j) \Rightarrow g_0 \models NAC_{L_0}$ .
- $(\Leftarrow)$  Let  $g_0 \not\models NAC_{L_0}$  with  $NAC_{L_0} = \{NAC(n_j) | j \in J\}$ . Then for some  $j \in J$  a morphism  $q : N_j \rightarrow G_0 \in \mathcal{Q}$  exists such that  $q \circ n_j = g_0$ . Let  $(e^*, m^*)$  be an epi- $\mathcal{M}$ -factorization of  $g_c$ . Construct  $X$  with  $p_1 : X \rightarrow E$  and  $m_1 : X \rightarrow N_j$  as

pullback of  $m^*$  and  $q$ .



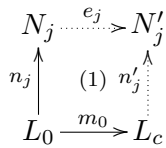
Then we have  $m_1 \in \mathcal{M}$  and  $p_1 \in \mathcal{Q}$ , since  $m^* \in \mathcal{M}$  and  $q \in \mathcal{Q}$ , PBs preserve  $\mathcal{M}$  and PBs along  $\mathcal{M}$  preserve  $\mathcal{Q}$ . Now construct  $Y$  with  $m_2 : E \rightarrow Y$  and  $p_2 : N_j \rightarrow Y$  as pushout of  $m_1$  and  $p_1$ . Then we have  $m_2 \in \mathcal{M}$ ,  $p_2 \in \mathcal{Q}$ , since  $m_1 \in \mathcal{M}$ ,  $p_1 \in \mathcal{Q}$ , POs preserve  $\mathcal{M}$  and POs along  $\mathcal{M}$  preserve  $\mathcal{Q}$ . Because of the induced PB-PO property the induced morphism  $x : Y \rightarrow G_0$  with  $x \circ m_2 = m^*$  and  $x \circ p_2 = q$  is a monomorphism in  $\mathcal{Q}$ .



It holds moreover that  $p_2, m_2 \circ e^*$  jointly epimorphic because  $e^*$  epimorphic and  $p_2, m_2$  jointly epimorphic. Summarizing we have the following equations:  $x \circ m_2 \circ e^* \circ m_0 = m^* \circ e^* \circ m_0 = g_c \circ m_0 = g_0 = q \circ n_j = x \circ p_2 \circ n_j$  and since  $x$  mono we have  $m_2 \circ e^* \circ m_0 = p_2 \circ n_j$ . Since  $m_2 \circ e^*$  and  $p_2$  are jointly epimorphic,  $p_2 \circ n_j = (m_2 \circ e^*) \circ m_0$  and  $p_2 \in \mathcal{Q}$  we can conclude that  $m_2 \circ e^* : L_c \rightarrow Y$  equals one of the morphisms  $n'_i : L_c \rightarrow N'_i \in D_{m_0}(NAC(n_j))$ . Moreover since  $x \circ m_2 \circ e^* = m^* \circ e^* = g_c$  and  $x \in \mathcal{Q}$  it holds that  $g_c \notin NAC(m_2 \circ e^*) = NAC(n'_i)$  and consequently  $g_c \notin NAC(n'_i) \Rightarrow g_c \notin D_{m_0}(NAC(n_j)) \Rightarrow g_c \notin D_{m_0}(NAC_{L_0})$ .  $\square$

**Remark 5.3** It is possible to cancel the fact that  $\mathcal{Q}$  is a set of special morphisms and thus generalize the definition of NAC-satisfiability. We should assume in this case though either that the NAC-morphism is in  $\mathcal{M}$  or each match is in  $\mathcal{M}$ . For this case now we can use a different more simple downward translation of NACs onto the LHS of the concurrent rule expressed by the following definition and lemma.

**Definition 5.4** [construction of NACs on  $L_c$  from NACs on  $L_0$  for general  $\mathcal{Q}$ ] Consider the following diagram:



For each  $NAC(n_j)$  on  $L_0$  with  $n_j : L_0 \rightarrow N_j \in \mathcal{M}$  and  $m_0 : L_0 \rightarrow L_c$ , construct pushout (1) with  $n'_j : L_c \rightarrow N'_j$  and  $e_j : N_j \rightarrow N'_j$  then

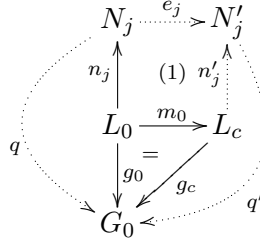
$$D_{m_0}(NAC(n_j)) = NAC(n'_j)$$

For each set of NACs  $NAC_{L_0} = \{NAC(n_j) | j \in J\}$  on  $L_0$  the downward translation of  $NAC_{L_0}$  is then defined as:

$$D_{m_0}(NAC_{L_0}) = \cup_{j \in J} D_{m_0}(NAC(n_j))$$

**Lemma 5.5 (equivalence of NACs on  $L_c$  and NACs on  $L_0$  for general  $\mathcal{Q}$ )**

Given  $g_c : L_c \rightarrow G_0$ ,  $m_0 : L_0 \rightarrow L_c$  with  $NAC_{L_0}$  and  $g_0 = g_c \circ m_0$  as in the following diagram:



then the following holds :

$$g_0 \models NAC_{L_0} \Leftrightarrow g_c \models D_{m_0}(NAC_{L_0})$$

**Proof**

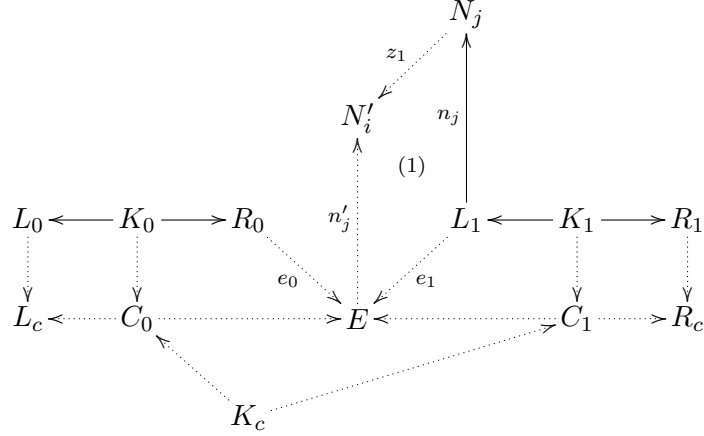
- $\Rightarrow$  Let  $g_c \not\models D_{m_0}(NAC(L_0)) = \cup_{j \in J} D_{m_0}(NAC(n_j))$  with  $NAC_{L_0} = \{NAC(n_j) | j \in J\}$ . Then for some  $j \in J$  there exists a morphism  $q' : N'_j \rightarrow G_0$  with  $q' \circ n'_j = g_c$ . It follows that  $q' \circ e_j \circ n_j = q' \circ n'_j \circ m_0 = g_c \circ m_0 = g_0$  and therefore  $q' \circ e_j$  violates  $NAC(n_j)$  for the match  $g_0$  which is a contradiction.
- $\Leftarrow$  Let  $g_0 \not\models NAC_{L_0}$  then for some  $j \in J$  there exists a morphism  $q : N_j \rightarrow G_0$  with  $q \circ n_j = g_0$ . Since (1) is a pushout and  $q \circ n_j = g_0 = g_c \circ m_0$  there exists a morphism  $q' : N'_j \rightarrow G_0$  with  $q' \circ e_j = q$  and  $q' \circ n'_j = g_c$ . Because of the last equation  $g_c \not\models D_{m_0}(NAC_{L_0})$  and this is a contradiction.

□

In Definition 2.9 and Lemma 2.11 it is explained how to construct an equivalent set of left NACs from a set of right NACs on a rule. Now we are ready to define a set of equivalent NACs on the LHS of the concurrent rule of a two-step transformation from the set of NACs on the LHS of the second rule of this transformation.

**Definition 5.6** [construction of NACs on  $L_c$  from NACs on  $L_1$ ] Given an E-dependency relation  $(e_0, e_1) \in \mathcal{E}'$  for the rules  $p_0$  and  $p_1$  and  $p_c = p_0 *_E p_1 : L_c \leftarrow$

$K_c \rightarrow R_c$  the E-concurrent rule of  $p_0$  and  $p_1$  as depicted in the following diagram:



For each  $NAC(n_j)$  on  $L_1$  with  $n_j : L_1 \rightarrow N_j$ :

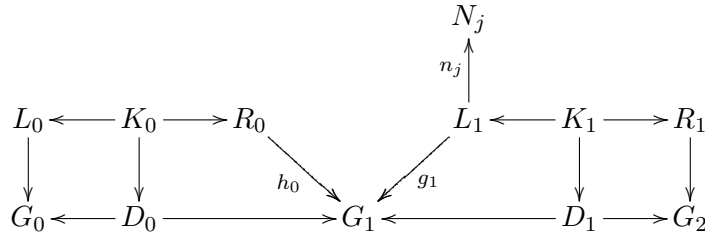
$$DL_{p_c}(NAC(n_j)) = L_p(D_{e_1}(NAC(n_j)))$$

with  $p : L_c \leftarrow C_0 \rightarrow E$  and  $D_{e_1}, L_p$  according to Def. 5.1 and Def. 2.9.

For each set of NACs  $NAC_{L_1} = \{NAC(n_j) | j \in J\}$  on  $L_1$  the down- and leftward translation of  $NAC_{L_1}$  is defined as:

$$DL_{p_c}(NAC_{L_1}) = \cup_{j \in J} DL_{p_c}(NAC(n_j))$$

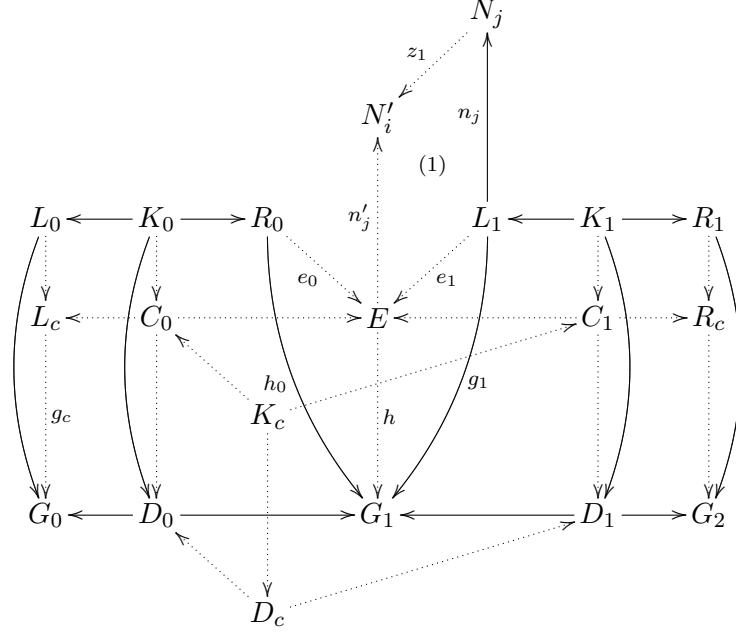
**Lemma 5.7 (equivalence of NACs on rule  $p_1$  and NACs on  $p_c$ )** *Given a two-step E-related transformation via  $p_0 : L_0 \leftarrow K_0 \rightarrow R_0$  and  $p_1 : L_1 \leftarrow K_1 \rightarrow R_1$*



with  $g_c$  being the match from the LHS of the E-concurrent rule  $p_c = p_0 *_E p_1$  into  $G_0$  (as described in the synthesis construction of Theorem 5.23 in [3]) then the following holds:

$$g_1 \models NAC_{L_1} \Leftrightarrow g_c \models DL_{p_c}(NAC_{L_1}).$$

**Proof** Consider the following diagram:



The E-concurrent rule  $p_0 *_{\mathcal{E}} p_1$  is the rule  $p_c : L_c \leftarrow K_c \rightarrow R_c$ , as described in Definition 5.21 in [3]. The derived span of the E-concurrent transformation  $G_0 \xrightarrow{p_c, g_c} G_2$  is  $G_0 \leftarrow D_c \rightarrow G_2$ . Because of Lemma 5.2  $g_1 \models NAC_{L_1} \Leftrightarrow h \models D_{e_1}(NAC_{L_1})$ . Moreover because of Lemma 2.9  $g_c \models L_p(D_{e_1}(NAC_{L_1})) \Leftrightarrow h \models D_{e_1}(NAC_{L_1})$  with  $p : L_c \leftarrow C_0 \rightarrow E$ . Note that  $L_p(D_{e_1}(NAC_{L_1})) = \cup_{j \in J} L_p(D_{e_1}(NAC(n_j))) = \cup_{j \in J} DL_{p_c}(NAC(n_j))$ . Consequently, it holds that  $g_1 \models NAC_{L_1} \Leftrightarrow g_c \models DL_{p_c}(NAC_{L_1})$ .  $\square$

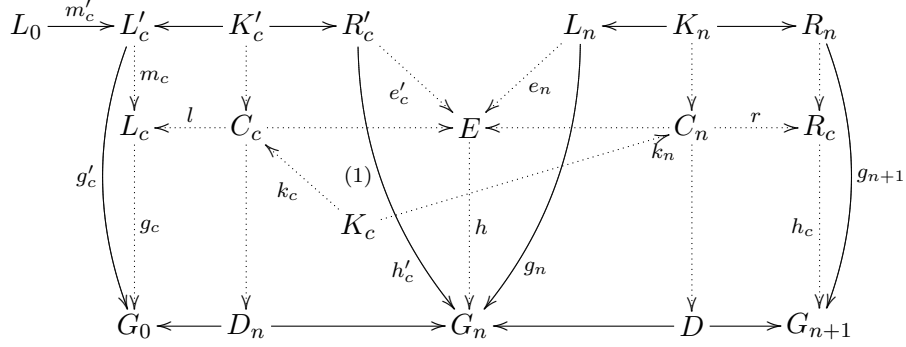
**Definition 5.8** [concurrent rule with NAC, concurrent (co-, lhs-)match induced by  $G_0 \xrightarrow{n+1} G_{n+1}$ ]

$n = 0$  For a direct transformation  $G_0 \Rightarrow G_1$  via match  $g_0 : L_0 \rightarrow G_0$ , comatch  $g_1 : R_1 \rightarrow G_1$  and rule  $p_0 : L_0 \leftarrow K_0 \rightarrow R_0$  with  $NAC_{p_0}$  the *concurrent rule*  $p_c$  with  $NAC$  induced by  $G_0 \Rightarrow G_1$  is defined by  $p_c = p_0$  with  $NAC_{p_c} = NAC_{p_0}$ , the *concurrent comatch*  $h_c$  is defined by  $h_c = g_1$ , the *concurrent lhs-match* by  $id : L_0 \rightarrow L_0$  and the *concurrent match*  $g_c$  by  $g_c = g_0 : L_0 \rightarrow G_0$ .

$n \geq 1$  Consider  $p'_c : L'_c \leftarrow K'_c \rightarrow R'_c$  (resp.  $g'_c : L'_c \rightarrow G_0$ ,  $h'_c : R'_c \rightarrow G_n, m'_c : L_0 \rightarrow L'_c$ ), the concurrent rule with NACs (resp. concurrent match, comatch, lhs-match) induced by  $G_0 \xrightarrow{n} G_n$ . Let  $((e'_c, e_n), h)$  be the  $\mathcal{E}' - \mathcal{M}'$  pair factorization of the comatch  $h'_c$  and match  $g_n$  of  $G_n \Rightarrow G_{n+1}$ . According to Fact 5.29 in [3] PO-PB decomposition, PO composition and decomposition lead to the diagram



below in which (1) is a pullback and all other squares are pushouts:



For a transformation sequence  $G_0 \xRightarrow{n+1} G_{n+1}$  the *concurrent rule*  $p_c$  with NACs (resp. concurrent match, comatch, lhs-match) induced by  $G_0 \xRightarrow{n+1} G_{n+1}$  is defined by  $p_c = L_c \xleftarrow{lo k_c} K_c \xrightarrow{ro k_n} R_c$  ( $g_c : L_c \rightarrow G_0$ ,  $h_c : R_c \rightarrow G_{n+1}$ ,  $m_c \circ m'_c : L_0 \rightarrow L_c$ ). Thereby  $NAC_{p_c}$  is defined by  $NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$ .

- Theorem 5.9 (Concurrency Theorem with NACs)** (i) *Synthesis.* Given a transformation sequence  $t : G_0 \xRightarrow{*} G_{n+1}$  via a sequence of rules  $p_0, p_1, \dots, p_n$ , then there is a synthesis construction leading to a direct transformation  $G_0 \Rightarrow G_{n+1}$  via the concurrent rule  $p_c : L_c \leftarrow K_c \rightarrow R_c$  with  $NAC_{p_c}$ , match  $g_c : L_c \rightarrow G_0$  and comatch  $h_c : R_c \rightarrow G_{n+1}$  induced by  $t : G_0 \xRightarrow{*} G_{n+1}$ .
- (ii) *Analysis.* Given a direct transformation  $G'_0 \Rightarrow G'_{n+1}$  via the concurrent rule  $p_c : L_c \leftarrow K_c \rightarrow R_c$  with  $NAC_{p_c}$  induced by  $t : G_0 \xRightarrow{*} G_{n+1}$  via a sequence of rules  $p_0, p_1, \dots, p_n$  then there is an analysis construction leading to a transformation sequence  $t' : G'_0 \xRightarrow{*} G'_{n+1}$  with NACs via  $p_0, p_1, \dots, p_n$ .
- (iii) *Bijective Correspondence.* The synthesis and analysis constructions are inverse to each other up to isomorphism.

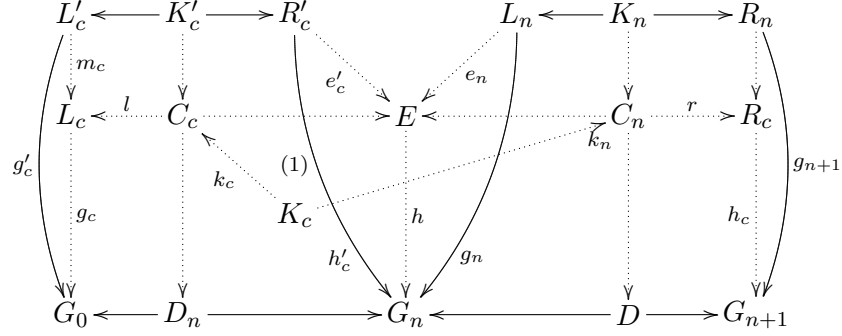
**Proof** We prove this theorem by induction over the number of transformation steps  $n + 1$ .

(i) *Synthesis.*

Basis.  $n=0$ . For a direct transformation  $t : G_0 \xRightarrow{p_0, g_0} G_1$  via match  $g_0 : L_0 \rightarrow G_0$  and rule  $p_0 : L_0 \leftarrow K_0 \rightarrow R_0$  with  $NAC_{p_0}$  the *concurrent rule*  $p_c$  with NAC induced by  $G_0 \Rightarrow G_1$  is defined by  $p_c = p_0$  with  $NAC_{p_c} = NAC_{p_0}$  and the *concurrent match*  $g_c$  is defined by  $g_c = g_0 : L_0 \rightarrow G_0$ . Therefore the synthesis construction is equal to  $G_0 \xRightarrow{p_c, g_c} G_1$ .

Induction Step. Consider  $t : G_0 \xRightarrow{n} G_n \Rightarrow G_{n+1}$  via the rules  $p_0, p_1, \dots, p_n$ . Let  $p'_c : L'_c \leftarrow K'_c \rightarrow R'_c$  (resp.  $g'_c : L'_c \rightarrow G_0$ ,  $h'_c : R'_c \rightarrow G_n$ ), be the concurrent rule with NACs (resp. concurrent match, comatch) induced by  $G_0 \xRightarrow{n} G_n$ . Suppose that  $G_0 \xRightarrow{p'_c, g'_c} G_n$  is a direct transformation with NAC leading to  $G_n$ . Let  $((e'_c, e_n), h)$  be the  $\mathcal{E}' - \mathcal{M}'$  pair factorization of the comatch  $h'_c$  and match  $g_n$  of  $G_n \Rightarrow G_{n+1}$ . PO-PB decomposition, PO composition and decomposition as described in Fact 5.29 in [3] lead to the diagram

below in which (1) is a pullback and all other squares are pushouts:



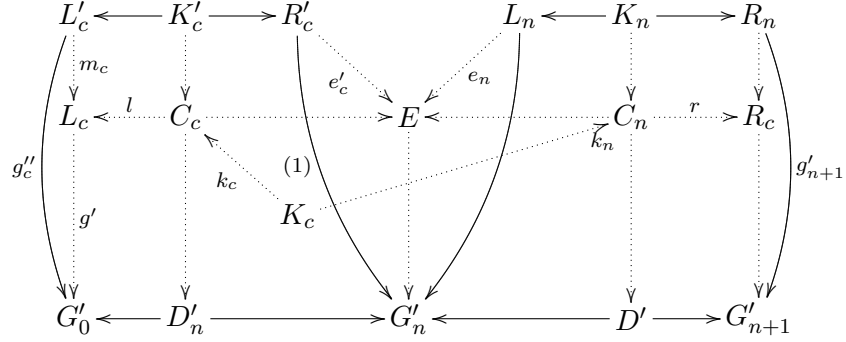
The concurrent rule  $p_c$  with NACs (resp. concurrent match, comatch) induced by  $G_0 \xrightarrow{n+1} G_{n+1}$  is  $p_c = L_c \xleftarrow{loke} K_c \xrightarrow{rokn} R_c$  ( $g_c : L_c \rightarrow G_0$ ,  $h_c : R_c \rightarrow G_{n+1}$ ). Thereby  $NAC_{p_c}$  is  $NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$ . We should prove that  $G_0 \xrightarrow{p_c, g_c} G_{n+1}$  is a valid direct transformation with NACs. At first an analogous synthesis construction to the one for two direct transformations without NACs in Theorem 5.23 in [3] can be done. Thus, in a second step we shall show that  $g_c$  satisfies  $NAC_{p_c}$  if  $g'_c$  satisfies  $NAC_{p'_c}$  and  $g_n$  satisfies  $NAC_{p_n}$ . This follows because of Lemma 5.2, Lemma 5.7 and the fact that  $G_0 \xrightarrow{p'_c, g'_c} G_n$  is a direct transformation via the rule  $p'_c$  with concurrent NAC  $NAC_{p'_c}$ .

(ii) *Analysis.*

Basis.  $n=0$ . For a direct transformation  $G'_0 \Rightarrow G'_1$  via the concurrent rule  $p_c = p_0$  with  $NAC_{p_c} = NAC_{p_0}$  the analysis construction is equal to  $G'_0 \Rightarrow G'_1$ .

Induction Step. Given a direct transformation  $G'_0 \Rightarrow G'_{n+1}$  via the concurrent rule  $p_c : L_c \leftarrow K_c \rightarrow R_c$  with  $NAC_{p_c}$  induced by  $t : G_0 \xRightarrow{*} G_{n+1}$  via a sequence of rules  $p_0, p_1, \dots, p_n$ . The concurrent rule  $p_c$  induced by  $t$  can be interpreted as  $p'_c *_{E_n} p_n$  in which the  $E_n$ -dependency relation between the rules is induced by the  $\mathcal{E}' - \mathcal{M}'$  pair factorization of the comatch  $h'_c$  induced by  $G_0 \xRightarrow{*} G_n$  and the match  $g_n$  of  $G_n \rightarrow G_{n+1}$  as described in Def. 5.8. So we have a direct transformation  $G'_0 \Rightarrow G'_{n+1}$  via  $p_c = p'_c *_{E_n} p_n$  and because of the Analysis part of Theorem 5.23 in [3] there is an analysis construction leading to a transformation sequence without NACs  $G'_0 \Rightarrow G'_n \Rightarrow G'_{n+1}$  via

$p'_c$  and  $p_n$  and matches  $g''_c$  resp.  $g'_n$ .



We know by assumption that the match  $g'$  of  $G'_0 \Rightarrow G'_{n+1}$  satisfies  $NAC_{p_c}$ . Since Lemma 5.2 and Lemma 5.7 hold in both directions, i.e. translate NACs in an equivalent way, we can conclude that  $NAC_{p'_c}$  and  $NAC_{p_n}$  are satisfied by  $g''_c$  resp.  $g'_n$ . Therefore  $G'_0 \Rightarrow G'_n \Rightarrow G'_{n+1}$  is a valid transformation sequence with NACs. Because of the induction hypothesis there exists an analysis construction  $G'_0 \Rightarrow G'_1 \Rightarrow \dots G'_n$  via  $p_0, p_1, \dots, p_{n-1}$  for  $G'_0 \Rightarrow G'_n$  via  $p'_c$ . Thus we obtain a transformation sequence with NACs  $G'_0 \Rightarrow G'_1 \Rightarrow \dots G'_{n+1}$  via  $p_0, p_1, \dots, p_n$  for the direct transformation  $G'_0 \Rightarrow G'_{n+1}$  via the concurrent rule  $p_c : L_c \leftarrow K_c \rightarrow R_c$  with  $NAC_{p_c}$ .

- (iii) *Bijective Correspondence.* The bijective correspondence follows from the fact that the  $\mathcal{E}' - \mathcal{M}'$  pair factorization is unique, and pushout and pullback constructions are unique up to isomorphism.

□

## 6 Embedding and Confluence

In the last section we have defined the concurrent rule  $p_c$  with NACs (resp. match  $g_c$ ) induced by a transformation  $t$ . We need this definition in order to define NAC-consistency which is an extra condition needed on top of boundary consistency to generalize the Embedding and Extension Theorem to transformations with NACs. Having generalized the notion of critical pairs, completeness, embedding and extension to transformations with NACs, it is now possible to formulate a sufficient condition on the critical pairs with NACs in order to obtain local confluence of an adhesive HLR system with NACs, i.e. to formulate the Critical Pair Lemma with NACs.

We start though with the definition of NAC-consistency for an extension morphism  $k_0$  w.r.t. a transformation  $t$ . It expresses that the extended concurrent match induced by  $t$  should fulfill the concurrent NAC induced by  $t$ .

**Definition 6.1** [NAC-consistency] A morphism  $k_0 : G_0 \rightarrow G'_0$  is called NAC-consistent w.r.t. a transformation  $t : G_0 \xrightarrow{*} G_n$  if  $k_0 \circ g_c \models NAC_{p_c}$  with  $NAC_{p_c}$  the concurrent NAC and  $g_c$  the concurrent match induced by  $t$ .

The Embedding Theorem for rules with NACs needs as extra condition on the extension morphism  $k_0$  NAC-consistency. Note the following renaming in order to

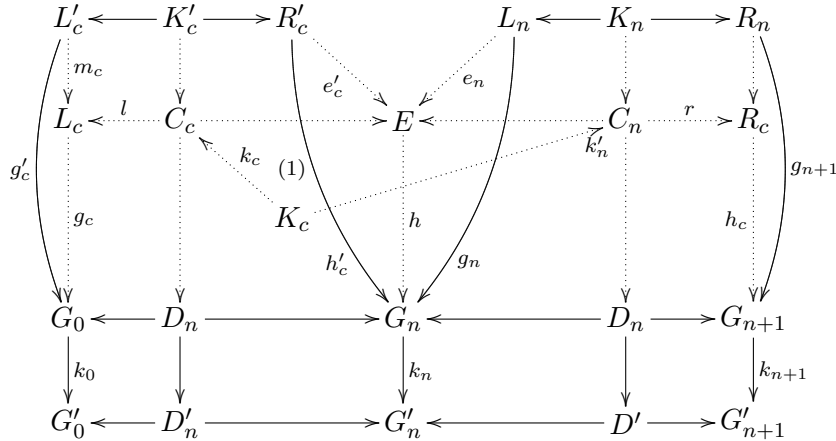
be able to distinguish better NAC-consistency from the consistency needed for the Embedding Theorem without NACs. In the following we speak about *boundary consistency* when we mean *consistency* as in Def. 6.12 of [3].

**Theorem 6.2 (Embedding Theorem with NACs)** *Given a transformation  $t : G_0 \xRightarrow{n} G_n$  with NACs. If  $k_0 : G_0 \rightarrow G'_0$  is boundary consistent and NAC-consistent w.r.t.  $t$  then there exists an extension diagram with NACs over  $t$  and  $k_0$ .*

**Proof** We prove this theorem by induction over the number of direct transformation steps  $n$ .

**Basis.**  $n=1$ . Consider a direct transformation  $t : G_0 \xRightarrow{p_0, g_0} G_1$  via match  $g_0 : L_0 \rightarrow G_0$  and rule  $p_0 : L_0 \leftarrow K_0 \rightarrow R_0$  with  $NAC_{p_0}$  and extension morphism  $k_0 : G_0 \rightarrow G'_0$ . Because of NAC-consistency  $k_0 \circ g_0 \models NAC_{L_0}$ . This means that the extension diagram over  $k_0$  and  $t$  without NACs as described in Theorem 6.14 in [3] is also an extension diagram over  $k_0$  and  $t$  with NACs.

**Induction Step.** Consider  $t : G_0 \xRightarrow{n} G_n \xRightarrow{p_n, g_n} G_{n+1}$  via the rules  $p_0, p_1, \dots, p_n$ . Let  $p'_c : L'_c \leftarrow K'_c \rightarrow R'_c$  (resp.  $g'_c : L'_c \rightarrow G_0$ ,  $h'_c : R'_c \rightarrow G_n$ ) be the concurrent rule with NACs (resp. concurrent match, comatch) induced by  $G_0 \xRightarrow{n} G_n$ . The induction hypothesis says that there exists an extension diagram with NACs over  $t' : G_0 \xRightarrow{n} G_n$  and  $k_0 : G_0 \rightarrow G'_0$ . This means in particular that  $k_0 \circ g_0 \models NAC_{L_0}, k_1 \circ g_1 \models NAC_{L_1}, \dots, k_{n-1} \circ g_{n-1} \models NAC_{L_{n-1}}$ , i.e. each extended match of the extension diagram satisfies the NACs on the corresponding rule. Moreover let  $G'_0 \leftarrow D'_n \rightarrow G'_n$  be the derived span of the extension diagram  $G'_0 \xRightarrow{n} G'_n$  over  $t$  and  $k_0$ . In the proof of Theorem 6.14 in [3] it is described how to obtain an extension diagram without NACs over  $t : G_0 \xRightarrow{n} G_n \Rightarrow G_{n+1}$  and  $k_0 : G_0 \rightarrow G'_0$ . The same construction can be made since  $k_0$  is boundary consistent. Now we still have to prove that the last extended match  $k_n \circ g_n$  of the extension diagram without NACs satisfies the set of NACs on the last rule  $p_n$  of the transformation sequence. Let  $((e'_c, e_n), h)$  be the  $\mathcal{E}' - \mathcal{M}'$  pair factorization of the comatch  $h'_c$  and match  $g_n$  of  $G_n \Rightarrow G_{n+1}$ . PO-PB decomposition, PO composition and decomposition lead to the diagram below as described in Fact 5.29 in [3] in which (1) is a pullback and all other squares are pushouts:



The concurrent rule  $p_c$  with NACs (resp. concurrent match, comatch) induced by

$G_0 \xRightarrow{n+1} G_{n+1}$  is  $p_c = L_c \xleftarrow{lok_c} K_c \xrightarrow{rok'_n} R_c$  ( $g_c : L_c \rightarrow G_0$ ,  $h_c : R_c \rightarrow G_{n+1}$ ). Thereby  $NAC_{p_c}$  is  $NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$  and  $G_0 \xRightarrow{p_c, g_c} G_n$ . Because of NAC-consistency of  $k_0$  we know that  $k_0 \circ g_c \models NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$ . Now because of Lemma 5.7 it follows that  $k_n \circ g_n \models NAC_{L_n}$  and thus we have an extension diagram with NACs over  $t$  and  $k_0$ .  $\square$

The following Extension Theorem with NACs describes the fact that boundary and NAC-consistency are not only sufficient, but also necessary conditions for the construction of extension diagrams for transformations with NACs.

**Theorem 6.3 (Extension Theorem with NACs)** *Given a transformation  $t : G_0 \xRightarrow{n} G_n$  with NACs with a derived span  $der(t) = (G_0 \xleftarrow{d_0} D_n \xrightarrow{d_n} G_n)$  and an extension diagram (1) as in the following picture:*

$$\begin{array}{ccccc} B & \xrightarrow{b_0} & G_0 & \xRightarrow{t} & G_n \\ \downarrow & (2) \downarrow k_0 & \downarrow & (1) \downarrow k_n & \\ C & \longrightarrow & G'_0 & \xRightarrow{t'} & G'_n \end{array}$$

then

- $k_0 : G_0 \rightarrow G'_0$  is boundary consistent w.r.t.  $t$ , with the morphism  $b : B \rightarrow D_n$ .
- $k_0 : G_0 \rightarrow G'_0$  is NAC-consistent w.r.t.  $t$ .
- Let  $p_c$  (resp.  $g_c$ ) be the concurrent rule with  $NAC_{p_c}$  (resp. concurrent match) induced by  $t$ . There is a direct transformation  $G'_0 \Rightarrow G'_n$  via  $der(t)$  with  $NAC_{der(t)} = D_{g_c}(NAC_{p_c})$  and match  $k_0$  given by pushouts (3) and (4) with  $h, k_n \in \mathcal{M}'$ .
- There are initial pushouts (5) and (6) over  $h \in \mathcal{M}'$  and  $k_n \in \mathcal{M}'$ , respectively, with the same boundary-context morphism  $B \rightarrow C$ .

$$\begin{array}{ccccc} G_0 & \xleftarrow{d_0} & D_n & \xrightarrow{d_n} & G_n \\ \downarrow k_0 & (3) & \downarrow h & (4) & \downarrow k_n \\ G'_0 & \longleftarrow & D'_n & \longrightarrow & G'_n \end{array} \quad \begin{array}{ccc} B & \xrightarrow{b} & D_n \\ \downarrow & (5) & \downarrow h \\ D'_n & \longrightarrow & D'_n \end{array} \quad \begin{array}{ccc} B & \xrightarrow{d_n \circ b} & D_n \\ \downarrow & (6) & \downarrow k_n \\ C & \longrightarrow & G'_n \end{array}$$

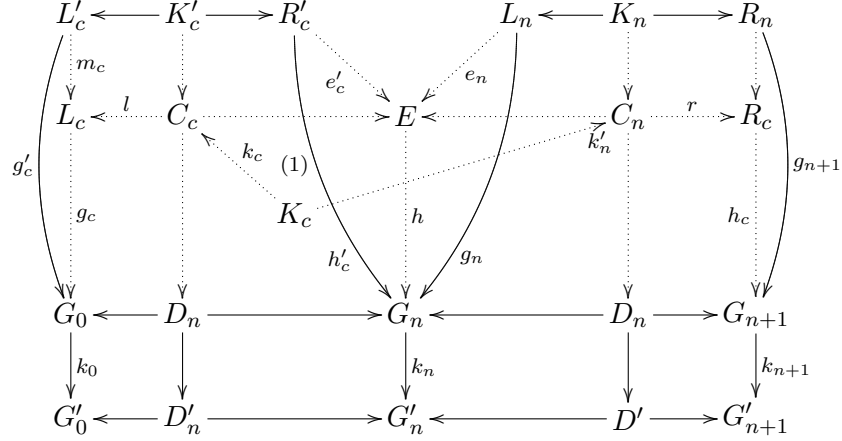
### Proof

- See proof of item 1 in Theorem 6.16 in [3].
- We should prove that  $k_0 \circ g_c$  with  $g_c$  the concurrent match induced by  $t$  satisfies  $NAC_{p_c}$  the concurrent NAC on the concurrent rule  $p_c$  induced by  $t$ . We prove this by induction over the number of direct transformation steps  $n$ .

**Basis.**  $n=1$ . Consider the extension diagram over the direct transformation  $t : G_0 \xRightarrow{p_0, g_0} G_1$  (via match  $g_0 : L_0 \rightarrow G_0$  and rule  $p_0 : L_0 \leftarrow K_0 \rightarrow R_0$  with  $NAC_{p_0}$ ) and extension morphism  $k_0 : G_0 \rightarrow G'_0$ . Because of Def. 4.2  $k_0 \circ g_0 \models NAC_{L_0}$  and therefore  $k_0$  is NAC-consistent w.r.t.  $t$ .

**Induction Step.** Consider the extension diagram over  $t : G_0 \xRightarrow{n} G_n \xRightarrow{p_n, m_n} G_{n+1}$  (via the rules  $p_0, p_1, \dots, p_n$  with NACs) and the extension morphism  $k_0 : G_0 \rightarrow$

$G'_0$ . Let  $p'_c : L'_c \leftarrow K'_c \rightarrow R'_c$  (resp.  $g'_c : L'_c \rightarrow G_0, h'_c : R'_c \rightarrow G_n$ ) be the concurrent rule with NACs (resp. concurrent match, comatch) induced by  $G_0 \xRightarrow{n} G_n$ . The induction hypothesis says that  $k_0$  is NAC-consistent w.r.t.  $t' : G_0 \xRightarrow{n} G_n$ . This means in particular that  $k_0 \circ g'_c$  satisfies  $NAC_{p'_c}$ . We should prove now that also  $k_0 \circ g_c$  satisfies  $NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$  the concurrent NAC on the concurrent rule  $p_c$  induced by  $t$ . Because of Lemma 5.2 and the induction hypothesis we have that  $k_0 \circ g_c \models D_{m_c}(NAC_{L'_c}) = D_{m_c}(NAC_{p'_c})$ . Because of Def. 4.2  $k_n \circ g_n \models NAC_{L_n}$  and because of Lemma 5.7 therefore  $k_0 \circ g_c \models DL_{p_c}(NAC_{L_n})$ .



- In item 2 in Theorem 6.16 in [3] it is proven that there is a direct transformation  $G'_0 \Rightarrow G'_n$  without NACs via  $der(t)$  and  $k_0$  given by the pushouts (3) and (4) with  $h, k_n \in \mathcal{M}'$ . So we still have to prove that  $k_0 \models NAC_{der(t)} = D_{g_c}(NAC_{p_c})$ . This follows because of Lemma 5.2 and the fact that  $k_0 \circ g_c \models NAC_{p_c}$  as proven in the former item.
- See proof of item 3 in Theorem 6.16 in [3].

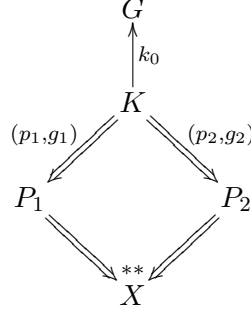
□

For the Critical Pair Lemma with NACs we need a stronger condition as in the case without NACs in order to obtain local confluence of the adhesive HLR system. In addition to strict confluence of the set of critical pairs we need also NAC-confluence. If a critical pair is strictly confluent via some transformations  $t_1$  and  $t_2$ , we call  $t_1$  and  $t_2$  a *strict solution* of the critical pair. NAC-confluence of a critical pair expresses that the NAC-consistency of an extension morphism w.r.t. a strict solution of the critical pair follows from the NAC-consistency of the extension morphism w.r.t. the critical pair itself.

**Definition 6.4** [strictly NAC-confluent] A critical pair  $P_1 \xleftarrow{p_1, g_1} K \xrightarrow{p_2, g_2} P_2$  is *strictly NAC-confluent* if and only if

- it is *strictly confluent* via some transformations  $t_1 : K \xrightarrow{p_1, g_1} P_1 \Rightarrow^* X$  and  $t_2 : K \xrightarrow{p_2, g_2} P_2 \Rightarrow^* X$
- and it is *NAC-confluent* for  $t_1$  and  $t_2$ , i.e. for every morphism  $k_0 : K \rightarrow G \in \mathcal{M}'$  which is NAC-consistent w.r.t.  $K \xrightarrow{p_1, g_1} P_1$  and  $K \xrightarrow{p_2, g_2} P_2$  it follows that  $k_0$  is

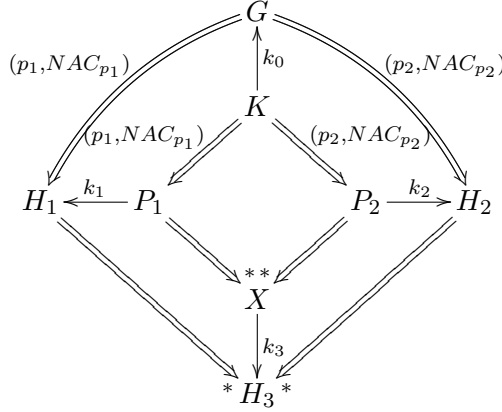
NAC-consistent w.r.t.  $t_1$  and  $t_2$ .



**Theorem 6.5 (Local Confluence Theorem - Critical Pair Lemma with NACs)**

*Given an adhesive HLR system with NACs, it is locally confluent if all its critical pairs are strictly NAC-confluent.*

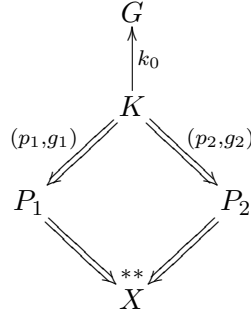
**Proof** If a pair of direct transformations is parallel independent then they are confluent because of the Local Church-Rosser Theorem. Suppose we have a pair of direct transformations  $H_1 \Leftarrow G \Rightarrow H_2$  in conflict. Because of Theorem 4.5 a critical pair  $P_1 \Leftarrow K \Rightarrow P_2$  exists, which can be embedded into this pair of direct transformations. We know that each critical pair is strictly NAC-confluent and consequently locally confluent. This means that an object  $X$  exists as in the following picture. We still have to prove though that  $H_1 \Rightarrow^* H_3 \Leftarrow^* H_2$ .



In the Critical Pair Lemma without NACs in [3] it is proven that the extension morphisms  $k_1$  and  $k_2$  are boundary consistent because of the fact that the critical pair  $P_1 \Leftarrow K \Rightarrow P_2$  is not only locally confluent, but also strictly confluent. Therefore extension diagrams without NACs over  $t_1 : K \Rightarrow P_1 \rightarrow^* X$  and  $t_2 : K \Rightarrow P_2 \Rightarrow^* X$  can be constructed s.t.  $G \Rightarrow H_1 \Rightarrow^* H_3$  and  $G \Rightarrow H_2 \Rightarrow^* H_3$ . The uniqueness of  $H_3$  is proven in [3]. We should still prove that the extended matches in these extension diagrams satisfy the NACs of the rule sequences in  $t_1$  and  $t_2$ . First we show that  $k_0$  is boundary consistent w.r.t.  $K \Rightarrow P_1$  (resp.  $K \Rightarrow P_2$ ). This is because we have an extension diagram over  $k_0$  and  $K \Rightarrow P_1$  (resp.  $K \Rightarrow P_2$ ) and Theorem 6.3. Moreover we have that  $k_0 \circ g_1 \models NAC_{p_1}$  (resp.  $k_0 \circ g_2 \models NAC_{p_2}$ ). This means that  $k_0$  is also NAC-consistent w.r.t. the direct transformation  $K \Rightarrow P_1$  (resp.  $K \Rightarrow P_2$ ). Since we have that the critical pair  $P_1 \Leftarrow K \Rightarrow P_2$  is strictly NAC-confluent it follows that

$k_0$  is NAC-consistent w.r.t  $t_1$  and  $t_2$ . Since  $k_0$  is NAC-consistent the Embedding Theorem for rules with NACs can be applied to  $k_0$ ,  $t_1$  and  $t_2$  to conclude that the extended matches in the extension diagrams over  $k_0$  and  $t_1$  (resp.  $t_2$ ) satisfy the NACs of the rule sequences in  $t_1$  (resp.  $t_2$ ).  $\square$

The following theorem formulates a sufficient condition for a critical pair  $P_1 \xleftarrow{p_1} K \xrightarrow{p_2} P_2$  which is strictly confluent via some transformations  $t_1 : K \xrightarrow{p_1, g_1} P_1 \Rightarrow^* X$  and  $t_2 : K \xrightarrow{p_2, g_2} P_2 \Rightarrow^* X$  to be also NAC-confluent for  $t_1$  and  $t_2$ . This means by definition that for every extension morphism  $k_0 : K \rightarrow G \in \mathcal{M}'$  which is NAC-consistent w.r.t.  $K \xrightarrow{p_1, g_1} P_1$  and  $K \xrightarrow{p_2, g_2} P_2$  it should follow that  $k_0$  is also NAC-consistent w.r.t.  $t_1$  and  $t_2$ . In the theorem two different conditions on each single  $NAC(n_{1,j})$  (resp.  $NAC(n_{2,j})$ ) of the concurrent NAC induced by transformation  $t_1$  (resp.  $t_2$ ) are given which lead to NAC-confluence if one of them is satisfied. The first condition expresses that there exists a suitable NAC on  $p_1$  (resp.  $p_2$ ) which evokes the satisfaction of  $NAC(n_{1,j})$  (resp.  $NAC(n_{2,j})$ ). The second condition first asks for a suitable morphism between the lhs's of the concurrent rules induced by both transformations  $t_1$  and  $t_2$ . Moreover it expresses that there exists a suitable NAC on  $p_2$  (resp.  $p_1$ ) which evokes the satisfaction of  $NAC(n_{1,j})$  (resp.  $NAC(n_{2,j})$ ).

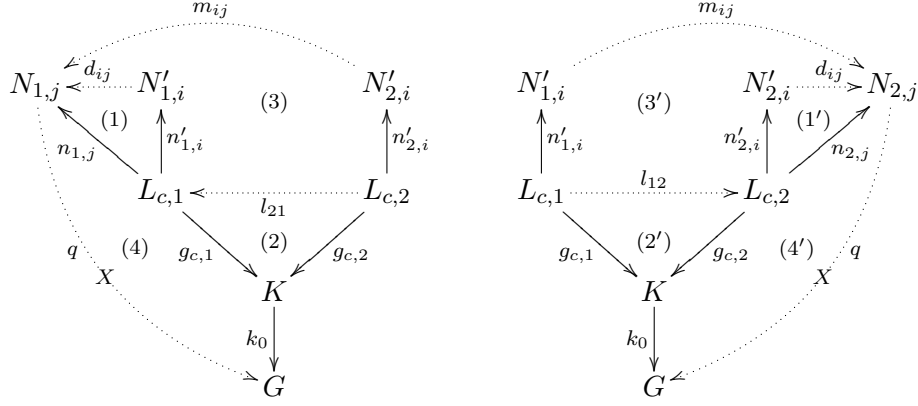


**Theorem 6.6 (Sufficient Condition for NAC-confluence)** *Given a critical pair  $P_1 \xleftarrow{p_1} K \xrightarrow{p_2} P_2$  which is strictly confluent via the transformations  $t_1 : K \xrightarrow{p_1, g_1} P_1 \Rightarrow^* X$  and  $t_2 : K \xrightarrow{p_2, g_2} P_2 \Rightarrow^* X$ . Let  $L_{c,1}$  (resp.  $L_{c,2}$ ) be the left-hand side of the concurrent rule  $p_{c,1}$  (resp.  $p_{c,2}$ ) and  $m_1 : L_1 \rightarrow L_{c,1}$  (resp.  $m_2 : L_2 \rightarrow L_{c,2}$ ) the lhs-match induced by  $t_1$  (resp.  $t_2$ ). Then the critical pair  $P_1 \xleftarrow{p_1} K \xrightarrow{p_2} P_2$  is also NAC-confluent for  $t_1$  and  $t_2$  and thus strictly NAC-confluent if one of the following conditions holds for each  $NAC(n_{1,j}) : L_{c,1} \rightarrow N_{1,j}$  (resp.  $NAC(n_{2,j}) : L_{c,2} \rightarrow N_{2,j}$ ) of the concurrent  $NAC_{p_{c,1}}$  induced by  $t_1$  (resp.  $NAC_{p_{c,2}}$  induced by  $t_2$ )*

- *there exists a  $NAC(n'_{1,i}) : L_{c,1} \rightarrow N'_{1,i}$  (resp.  $NAC(n'_{2,i}) : L_{c,2} \rightarrow N'_{2,i}$ ) in  $D_{m_1}(NAC_{p_1})$  (resp.  $D_{m_2}(NAC_{p_2})$ ) and a morphism  $d_{ij} \in \mathcal{Q} : N'_{1,i} \rightarrow N_{1,j}$  (resp.  $d_{ij} \in \mathcal{Q} : N'_{2,i} \rightarrow N_{2,j}$ ) such that (1) (resp. (1')) commutes.*
- *there exists a morphism  $l_{21} : L_{c,2} \rightarrow L_{c,1}$  (resp.  $l_{12} : L_{c,1} \rightarrow L_{c,2}$ ) s.t. (2) (resp. (2')) commutes and in addition a  $NAC(n'_{2,i}) : L_{c,2} \rightarrow N'_{2,i}$  (resp.  $n'_{1,i} : L_{c,1} \rightarrow N'_{1,i}$ ) in  $D_{m_2}(NAC_{p_2})$  (resp.  $D_{m_1}(NAC_{p_1})$ ) with a morphism  $m_{ij} : N'_{2,i} \rightarrow N_{1,j} \in \mathcal{Q}$  (resp.  $m_{ij} : N'_{1,i} \rightarrow N_{2,j} \in \mathcal{M}$ ) s.t.  $n_{1,j} \circ l_{21} = m_{ij} \circ n'_{2,i}$  (resp.*



$$n_{2,j} \circ l_{12} = m_{ij} \circ n'_{1,i}).$$



**Proof** We shall prove that  $k_0$  is NAC-confluent for  $t_1$  and  $t_2$ , i.e. each  $k_0 : K \rightarrow G \in \mathcal{M}'$  which is NAC-consistent w.r.t.  $K \xrightarrow{p_1} P_1$  and  $K \xrightarrow{p_2} P_2$  is also NAC-consistent w.r.t.  $t_1 : K \xrightarrow{p_1} P_1 \Rightarrow^* X$  and  $t_2 : K \xrightarrow{p_2} P_2 \Rightarrow^* X$ . Suppose that  $k_0 \circ g_{c,1}$  does not satisfy the concurrent  $NAC_{p_{c,1}}$  induced by  $t_1$ , i.e.  $k_0$  is not NAC-consistent w.r.t.  $t_1$ . Then we have a morphism  $q \in \mathcal{Q}$  of a NAC-object  $N_{1,j}$  of the concurrent  $NAC_{p_{c,1}}$  into  $G$  s.t. triangle (4) commutes, i.e.  $k_0 \circ g_{c,1} = q \circ n_{1,j}$ . Now one of the following two reasonings can be made:

- Because of the existence of a morphism  $d_{ij}$  such that (1) commutes, we have that  $k_0 \circ g_{c,1} = q \circ d_{ij} \circ n'_{1,i}$ . Moreover  $q \circ d_{ij}$  is a morphism in  $\mathcal{Q}$  because  $d_{ij}, q \in \mathcal{Q}$ . This means that the extended match  $k_0 \circ g_{c,1}$  does not satisfy  $D_{m_1}(NAC_{p_1})$ . Because of Lemma 5.2 now it follows that  $k_0$  is not NAC-consistent w.r.t.  $K \xrightarrow{p_1} P_1$  and this is a contradiction.
- Because of the existence of  $l_{21} : L_{c,2} \rightarrow L_{c,1}$  s.t. (2) commutes and  $m_{ij} : N_{2,i} \rightarrow N_{1,j}$  such that  $n_{1,j} \circ l_{21} = m_{ij} \circ n'_{2,i}$  the following equations hold:  $q \circ m_{ij} \circ n'_{2,i} = q \circ n_{1,j} \circ l_{21} = k_0 \circ g_{c,1} \circ l_{21} = k_0 \circ g_{c,2}$ . Now  $q \circ m_{ij} \in \mathcal{Q}$  because of the composition property and thus  $k_0 \circ g_{c,2}$  does not satisfy  $D_{m_2}(NAC_{p_2})$ . Because of Lemma 5.2 now it follows that  $k_0$  is not NAC-consistent w.r.t.  $K \xrightarrow{p_2} P_2$  which is a contradiction.

Analogously we can prove that each  $k_0 : K \rightarrow G$  which is NAC-consistent w.r.t.  $K \Rightarrow P_1$  and  $K \Rightarrow P_2$  is also NAC-consistent w.r.t.  $t_2$ .  $\square$

The following corollary follows directly from this theorem.

**Corollary 6.7** *A critical pair  $P_1 \xleftarrow{p_1} K \xrightarrow{p_2} P_2$  is strictly NAC-confluent if*

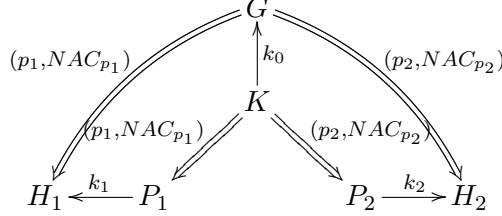
- *it is strictly confluent via the transformations  $t_1 : K \xrightarrow{p_1, g_1} P_1 \Rightarrow^* X$  (resp.  $t_2 : K \xrightarrow{p_2, g_2} P_2 \Rightarrow^* X$ ) and both  $P_1 \Rightarrow^* X$  and  $P_2 \Rightarrow^* X$  are transformation sequences without NACs.*

**Proof** In this case  $NAC_{p_{c,1}} = D_{m_1}(NAC_{p_1})$  and therefore  $d_{ij} = id$  for each single NAC in  $NAC_{p_{c,1}}$  such that the first condition of Theorem 6.6 always holds. Analogously we can argue for the case  $NAC_{p_{c,2}}$ .  $\square$

In order to formulate not only a sufficient, but also necessary condition for local

confluence of an adhesive HLRL system with NACs we define the so-called set of extended critical pairs.

**Definition 6.8** [extended critical pair] For each critical pair  $P_1 \Leftarrow K \Rightarrow P_2$ , which is not strictly NAC-confluent, an *extended critical pair* is a pair of direct transformations  $H_1 \Leftarrow G \Rightarrow H_2$  as in the following figure, with  $k_0 : K \rightarrow G \in \mathcal{M}'$  a NAC-consistent and boundary-consistent morphism w.r.t.  $K \Rightarrow P_1$  and  $K \Rightarrow P_2$ :



**Remark 6.9** If we take  $k_0 = id$ , then each critical pair which is not strictly NAC-confluent belongs to the set of its extended critical pairs.

**Theorem 6.10** *An adhesive AHLR system with NACs is locally confluent if and only if all its extended critical pairs are locally confluent.*

**Proof**

- $\Rightarrow$  If one of the extended critical pairs is not locally confluent, then the system is not locally confluent, since an extended critical pair is nothing but a pair of direct transformations.
- $\Leftarrow$  Let  $H_1 \Leftarrow G \Rightarrow H_2$  be a pair of direct transformations. If they are parallel independent, then they are locally confluent, because of the local Church-Rosser property. If they are not, then there exists a critical pair  $P_1 \Leftarrow K \Rightarrow P_2$  because of Theorem 4.5 which can be embedded into this pair of direct transformations. If this critical pair is strictly NAC-confluent, then  $H_1 \Leftarrow G \Rightarrow H_2$  is locally confluent as it is proven in Theorem 6.5. Suppose that the critical pair  $P_1 \Leftarrow K \Rightarrow P_2$  is not strictly NAC-confluent. Because of Theorem 6.3 we have that  $k_0 \in \mathcal{M}'$  is boundary- and NAC-consistent w.r.t.  $K \Rightarrow P_1$  and  $K \Rightarrow P_2$ . Then  $H_1 \Leftarrow G \Rightarrow H_2$  is an extended critical pair by definition and all extended critical pairs are locally confluent by assumption.

□

## 7 Conclusion

In this paper the most significant results are given in order to extend the theory on Algebraic Graph Transformation in [3] to Transformations with Negative Application Conditions. Summarizing the most important ones: these are Local Church-Rosser Theorem, Parallelism Theorem, Completeness Theorem, Concurrency Theorem, Embedding and Extension Theorem, and Local Confluence Theorem. These results for transformations with NACs are formulated in the context of the Adhesive High-Level Replacement Framework introduced in [3] with an extra

necessary morphism class  $\mathcal{Q}$ . The rather technical description of all theorems and proofs serves as a basis for further explanations and investigations on this subject together with illustrations of this new theory. Future work will be necessary on the applicability and refinement of all new results and on the development of efficient analysis algorithms for transformations with NACs.

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